Delegating Experiments

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Abstract

A principal wants information to help her decide whether to approve a project. She delegates costly experimentation to an agent, who wants her to approve the project and has private information about the state. The principal can influence experimentation only by restricting the experiments that the agent can undertake: she cannot commit to approval, and no transfers are possible. For example, the FDA may select a set of clinical trials that are acceptable for testing a new drug, but cannot pay drug companies or weaken its threshold for approval. We show the principal can screen the agent — and thus learn his private information — by offering a menu of experiments that differ in conditional expected payoffs across states. Doing so is always optimal: screening dominates pooling. Private information distorts the optimal menu by making the false negative rate inefficiently high. In our drug approval application, too many good drugs are rejected.

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1 Introduction

Individuals and organizations often delegate the acquisition of information to help them make decisions. Drug regulators delegate the conduct of clinical trials to pharmaceutical companies;[1] central banks delegate some components of stress testing to independent banks; and judicial systems delegate the investigation of defendants to police and prosecutors in criminal trials. Frequently, the agents that they delegate to have private information about the things they are asked to investigate, and care about the decisions that result from their investigations. For instance, pharmaceutical firms are better informed about the quality of their prototype drugs than the Food and Drug Administration (FDA), and want those drugs to be approved; retail banks are better informed than the Federal Reserve about their resilience to financial shocks, and want to pass stress tests.

This delegation problem is the primary concern of this paper. Unlike classic delegation problems, such as Holmström (1977) and Alonso and Matouschek (2008), where the principal delegates a decision to a privately informed agent, we consider settings where the principal retains decision making power but delegates the choice of an *experiment* to the agent. For instance, the FDA does not allow pharmaceutical companies to make an approval decision for a new drug, but instead requires them to conduct a clinical trial that must follow certain guidelines. This paper asks whether and how an agent can be induced to reveal his private information about the state by restricting the set of experiments that the principal will accept (e.g. through clinical trial guidance, rules for financial stress tests, or setting discovery rules in criminal trials). We show that they can, and moreover, that getting them to reveal their information (rather than letting it remain private) is al-

[1]The Food and Drug Administration (FDA) requires drug manufacturers to perform a wide variety of tests of quality prior to approving a new drug. First, drug manufacturers must test a proposed new compound on a variety of animal species to test if the compound is toxic. Then, drug companies submit an Investigational New Drug (IND) request, which includes information on the drug's compound. In the IND request, drug manufacturers outlay an experimentation plan going forward. They specify how they will test the drug on human subjects. Experimentation on human subjects proceeds in 4 stages. The first 3 of these stages (small scale tests measuring safety, medium scale tests measuring effectiveness, and large scale tests measuring interactions with other drugs) are lab-trials. Stage 4 involves the drug company periodically monitoring the drug after it goes to market. Throughout the procedure, the FDA's Center for Drug Evaluation and Research (CDER) carefully reviews the drug company's data and determines if the drug's anticipated benefit outweighs the risk. If so, the FDA approves the drug.

ways optimal.

We work in a model with two states of the world: a *low* state and a *high* state. A principal will decide whether to approve or reject a project, preferring to approve when the state is high. An agent prefers that the principal approves the project. The agent receives a private, noisy signal about the state. In order to persuade the principal to approve the project, the agent can conduct an experiment. Experiments produce hard information[2] and are costly, with constant marginal cost.[3] Consequently, experiments that fully reveal the state are prohibitively costly, and so are never chosen by the agent. The principal can restrict the agent to experiments from a *menu* designed by the principal. However, the agent cannot be compelled to experiment; they always have the option to not communicate.[4]

In settings like ours where transfers are not available, delegating experiments can benefit the principal relative to delegating the decision directly as in Holmström (1977) and Alonso and Matouschek (2008). For instance, in our setting, if the principal delegated the approval decision to the agent, the agent would always opt to approve the project. More generally, in any setting where an agent's payoffs from approval are independent of his type, a principal cannot induce the agent to reveal any private information merely by delegating the decision. However, since the agent's payoff from conducting a given experiment depends on his type, delegating experimentation can induce the agent to reveal his private information at the same time as he gathers additional information that is relevant to the approval decision.

The menu offered by the principal can either be *screening*, where each type selects a distinct experiment, or *pooling*, where some types select the same experiment. In canonical mechanism design settings, any pooling menu can be converted into an equivalent screening menu by simply "adding replicas" of objects in the menu.[5] However, in our setting, knowing the agent's type that conducts a particular experiment *changes* the distribution of posteriors it induces. Thus, because the principal cannot commit to approving the project, pooling menus cannot be converted into equivalent screening menus and must be treated separately. Choosing a screening menu instead of a pooling menu benefits the principal di-

[2] That is, the results of experimentation are verifiable and credible to any interested party.

[3] In the sense of Pomatto et al. (2023).

[4] That is, they can choose to "veto" a particular delegation set (Kartik et al., 2021).

[5] This is a consequence of the revelation principle.

rectly, because it allows her to learn the agent's private information. But there is an important trade-off: for some types, the agent's participation constraint is more difficult to satisfy in screening menus than in pooling menus. Theorem 1 shows that the benefit from learning the agent's private information outweighs the cost of a stricter participation constraint. Consequently, the optimal menu must be a screening menu. In the drug regulation setting, for instance, a regulator can produce better approval decisions by offering drug developers a menu of clinical trial protocols to choose from, rather than specifying a single protocol.

We show that it is possible to characterize implementable screening menus in a manner that is reminiscent of standard characterizations in the contracting and the literature on mechanism design with transfers,[6] despite the fact that the principal does not have access to transfers (Proposition 1). Importantly, implementable screening menus satisfy a monotonicity condition which allows us to intuitively describe the principal's screening procedure: the principal constructs a menu of experiments that differ in state-conditional expected payoffs for the agent. In any implementable screening menu, the difference between agent's expected payoff when the state is high and his expected payoff when the state is low is increasing in his type report. Since the agent's private information is precisely his belief regarding the state, the agent's payoff from increasing or decreasing approval probabilities in each state depends on his type. These values can vary separately, just like qualities and transfers in Mussa and Rosen (1978).

We next turn our attention to characterizing the principal's optimal screening menu. The principal's problem is one of constrained information design; we leverage the duality results of Doval and Skreta (2023) to show that it is the unique maximizer of a Lagrangian. This allows us to show that each of the experiments induced by the principal are binary and — just as in screening problems with transfers — maximize the difference of a surplus term and an information rent term.

However, because these terms contain Lagrange multipliers whose values are not pinned down, existing results characterizing distortion from asymmetric information (e.g., Maskin and Riley (1984), or Yoder (2022) when the choice is an experiment) do not immediately carry over to our setting. Moreover, since transfers are absent, distortion is not away from a single efficient experiment, but an entire Pareto frontier. We thus compare experiments from the optimal menu to *Pareto-*

[6] e.g., Myerson (1981), Mussa and Rosen (1978), Maskin and Riley (1984).

improving efficient experiments. We first arrive at the standard result from transferable utility contracting that there is *no distortion at the top*: the highest type's experiment is efficient, since their experiment does not appear in the local downward binding incentive constraints for any other type (Proposition 4). However, there is also *distortion everywhere else*: no other type conducts an efficient experiment (Theorem 3), and in particular, the principal may find it optimal to inefficiently exclude some types by inducing them to conduct a totally uninformative experiment.

This distortion does not necessarily take the form of a *less* informative experiment.[7] In particular, the false positive rate of an experiment induced by the principal may be higher or lower than that of an efficient experiment that Pareto improves upon it. But for types (other than the highest) that are not excluded by the principal, Theorem 3 shows that distortion always results in an inefficiently high false negative rate. In fact, the same is true relative to the take-it-or-leave-it offer that the principal would make to the agent if she knew his type (which is generally not a Pareto improvement): The experiment that the principal induces with asymmetric information has a higher false negative rate (Proposition 5). Hence, relative to either benchmark, the principal approves good projects (i.e., those for which the state is high) too infrequently.

Related Literature

Within the literature on information design (e.g., Kamenica and Gentzkow (2011)), other authors (e.g., Hedlund (2017); Kosenko (2023)) have explored the consequences of private information for the sender. Our paper innovates by allowing allowing another party (the principal) to constrain the researcher's experimentation decision. Essentially, we consider a screening problem, whereas previous contributions to this literature have focused on signaling problems.

Our paper belongs to a large body of work on delegation following Holmström (1977). Unlike many papers in this literature, the choice being delegated in our model has much higher dimension than the space of agent types. As we show, this gives the principal a greater ability to utilize the agent's private information: Alonso and Matouschek (2008) show that when delegating a one-dimensional choice, the principal can only choose a subset of types to force into a corner solution, while leaving the rest effectively unconstrained. But in our model, the princi-

[7] Unlike in Yoder (2022).

pal can fully screen the agent's private information (and in fact, we show doing so is always optimal) even without access to transfers. This feature is reminiscent of Koessler and Martimort (2012), who describe optimal delegation in settings where delegation sets are subsets of $\mathbb{R}^2$. They find that the *spread* between the decisions can allow the designer to screen the agent's private information. Analogous to spread in Koessler and Martimort (2012), we show that in our model, the designer can use the difference in the agent's expected utility conditional on the state being high and the agent's expected utility conditional on the state being low to screen the agent's private information.

Especially relevant to our paper is a recent literature on the delegation of *dynamic* experimentation. Guo (2016) studies a one-armed bandit model where an agent has private information about the risky arm's payoff, and the principal can limit the agent's freedom to allocate resources between arms (in the form of a history-dependent policy). Closer to our paper is McClellan (2022), who also considers a setting where an agent experiments to influence the approval decision of a principal. The key differences relative to our paper are that (a) the principal incentivizes the agent by committing to an approval rule, rather than limiting the experiments that the agent can conduct, and (b) because experimentation is dynamic á la Wald (1947), the principal faces additional incentive constraints.[8]

We also follow a large recent literature on contracting for flexible information acquisition. Rappoport and Somma (2017), Whitmeyer and Zhang (2022), and Sharma et al. (2024) each consider models where the principal does not possess private information, and so the contracting problem is one of moral hazard. They focus on the impact of risk aversion and limited liability constraints on the optimal payment scheme, as well as whether that scheme can be implemented by con-

[8]The Wald (1947) setup is closely related to flexible information acquisition with likelihood ratio costs; see Morris and Strack (2019). But because the principal only observes the ex post realization of the agent's experiment in McClellan (2022), he cannot fully restrict the design of the agent's experiment (or infer his private information from it) the way he can in our model. Conversely, because the principal cannot commit to an approval threshold in our model, he cannot use it as an instrument to screen the agent the way he can in McClellan (2022).

An earlier version (McClellan, 2017) considers the case of *two-sided commitment* where the agent can commit to an experimentation policy, and so the principal only faces a static incentive constraint. When the principal's inability to commit to an approval rule is not binding, our results also characterize the optimal mechanism in this setting: As Morris and Strack (2019) show, the cost of attaining a distribution of posteriors with Wald (1947) experimentation is just its LLR cost. Thus, a menu of static-threshold stopping mechanisms in McClellan's (2017) setting (as he shows is optimal with two-sided commitment) is equivalent to a menu of binary Blackwell experiments in ours.

tracting on the experiment's result and/or the realized state, rather than on the experimental protocol itself. Closer to this paper are Yoder (2022) and Wang (2023), who (like us) consider settings where the agent has private information, but (unlike us) where that private information concerns the agent's cost of experimentation. Relative to this literature, our paper has three novel features: (i) transfers are unavailable to the principal; (ii) the principal's decision is payoff-relevant to the agent; and (iii) the agent has private information about the state of the world itself.

This article is also related to work on the design of approval rules in statistical decision theory. Tetenov (2016) studies how a regulator can design a statistical test when agents have private information about the state. In his setting, the principal chooses a single experiment which all agents conduct, and commits to approval when the experiment produces a positive result. In contrast, our model highlights the usefulness of a menu of experiments to screen the agent. Jagadeesan and Viviano (2024) study the design of publication rules (i.e., should unsurprising results be published?) when agents decide how to experiment. In their setting, researchers do not possess ex-ante[9] private information, but can manipulate their findings.

2 Model

There is a binary state of the world $\omega \in \{0, 1\}$. An *agent* (he) with private but imperfect information about ω can publicly conduct a costly experiment about the state. A *principal* (she) can choose the set of experiments available to the agent. Before the agent observes his private information, both he and the principal place prior probability β_0 on the event $\omega = 1$.

The Principal After observing the experiment conducted by the agent and its result, the principal must decide whether to approve a project (such as a new drug application) whose value to her depends on the state of the world. She wants to approve the project in one state (without loss, $\omega = 1$) but not the other. Approving the project gives the principal a payoff of $w_1 > 0$ when $\omega = 1$ and $w_0 < 0$ when $\omega = 0$, while disapproval yields a payoff of zero.[10] Hence, when her belief is

[9] That is, they are not more informed about the state prior to choosing an experiment.

[10] Since the principal cannot make transfers to the agent, letting her disapproval payoff be constant at zero is without loss.

$P(\omega = 1) = \beta$, she chooses to approve the project if and only if[11]

$$\beta w_1 + (1-\beta) w_0 \geq 0 \Leftrightarrow \beta \geq \frac{-w_0}{w_1 - w_0} =: b.$$

We call b the principal's *threshold belief*, and assume that $\beta_0 < b < 1$: she is willing to approve if and only if she receives information that is sufficiently suggestive of state 1. Hence, when her belief is β, her interim expected payoff is given by

$$W(\beta) \equiv \begin{cases} 0, & \beta < b; \\ w_0 + (w_1 - w_0)\beta, & \beta \geq b. \end{cases}$$

Throughout, we assume that the principal cannot commit to an alternative decision rule in an attempt to incentivize the agent to reveal more information.

The Agent The agent privately observes a signal about ω that causes him to update his belief to $\theta \in \Theta := \{\theta_0, \dots, \theta_N\}$, where for each $1 \leq n \leq N$, $\theta_n > \theta_{n-1}$. We refer to the interim belief θ as the agent's *type*, and describe its distribution with the probability mass function σ; by Bayes' rule, $\sum_{n=1}^{N} \sigma(\theta_n)\theta_n = \beta_0$. We assume that $\theta_N < b$, so that no type would be able to convince the principal to approve the project with only his private signal.

The agent is motivated to conduct costly experiments by his desire to persuade the principal to approve the project. He has *transparent motives*, in the sense that his payoffs are state-independent: He receives a payoff of $u > 0$ if the project is approved, and zero otherwise. If the principal's posterior belief is β after observing the agent's experiment and its result, the agent's gross payoff[12] is $U(\beta)$, where

$$U(\beta) \equiv \begin{cases} 0, & \beta < b; \\ u, & \beta \geq b. \end{cases}$$

Experimentation The agent conducts a Blackwell experiment of the form $(S, \pi : \Omega \to \Delta(S))$, where S is a set of signal realizations.[13] We typically abuse notation and refer to this experiment as π when the set S is clear from context. Let Π de-

[11]To avoid existence issues, we assume that the principal approves the project when her belief is equal to the threshold b.

[12]Before considering the cost of the experiment.

[13]Naturally, S must be a Polish set.

note the set of Blackwell experiments.[14] When the result of an experiment is observed by an individual with belief α, π induces a distribution over posterior beliefs $\langle \pi|\alpha \rangle \in \Delta([0,1])$ through Bayes' rule, where $E_{\langle \pi|\alpha \rangle}\beta = \alpha$. We say that two experiments π and π' are *Blackwell equivalent*, denoted $\pi \sim_B \pi'$, if for any $\alpha \in [0,1]$, $\langle \pi|\alpha \rangle = \langle \pi'|\alpha \rangle$.[15] We denote the Blackwell ordering by $\succsim_B$ throughout.

Experimentation is costly for the agent. We assume that, when the agent conducts the experiment π, he pays the *log-likelihood ratio cost* (Pomatto et al., 2023) $C(\pi)$, where

$$C(\pi) \equiv \sum_{\omega \in \{0,1\}} \int_S \log\left(\frac{\pi(s|\omega)}{\pi(s|1-\omega)}\right) d\pi(s|\omega) = E_{\langle \pi|\alpha \rangle}[G(\beta|\alpha)] \text{ for any } \alpha \in (0,1),$$

where $G(\beta|\alpha) \equiv \frac{\beta}{\alpha}\log\left(\frac{\beta}{1-\beta}\right) + \frac{1-\beta}{1-\alpha}\log\left(\frac{1-\beta}{\beta}\right)$ is the *posterior-specific cost* for β given an initial belief α. We assume that these costs are not so great as to make experimentation unprofitable for the agent when the principal correctly infers his type: For all $\theta \in \Theta$, there exists $\pi \in \Pi$ such that $E_{\langle \pi|\theta \rangle}[U(\beta)] - C(\pi) > 0$.

The log-likelihood ratio (LLR) cost function has three desirable properties in the context of our model. First, LLR costs are *prior-independent*. For a fixed experiment $\pi \in \Pi$, the cost of conducting the experiment $C(\pi)$ does not depend on the agent's type. Experiments should be interpreted as physical processes in our model (e.g. clinical trials and stress tests), which is intuitively consistent with prior-independent information costs (Bloedel and Zhong, 2021). Second, LLR costs are posterior separable.[16] Posterior separability yields tractability; in information design involving posterior separable cost functions, the concavification approach in Kamenica and Gentzkow (2011) can be used to characterize solutions. Third, among prior-independent cost functions, LLR costs are uniquely characterized by a *constant marginal cost condition* (Pomatto et al., 2023): $C(\alpha\pi + (1-\alpha)\pi') = \alpha C(\pi) + (1-\alpha)C(\pi')$ for all $\alpha \in [0,1]$ and all experiments π, π'. That is, the cost of a composite experiment that randomizes over π and π' is the same as the ex-

[14]Formally, let $\Pi = \{(S,\pi) : S \in \mathcal{S}\}$ be the set of all Blackwell experiments where $\mathcal{S}$ is a sufficiently rich set containing sets of signal realizations. It suffices, for instance, to let $\mathcal{S}$ be any uncountable Polish space.

[15]Equivalently, $\pi \succsim_B \pi'$ and $\pi' \succsim_B \pi$ where $\succsim_B$ is the Blackwell informativeness ordering. When showing two experiments are Blackwell equivalent, it suffices to show that $\langle \pi|\alpha \rangle = \langle \pi'|\alpha \rangle$ for some non-degenerate prior $\alpha \in (0,1)$.

[16]But not uniformly posterior separable.

pected cost of randomizing over π and π'. This last feature (and the LLR functional form) are not needed for most of our main results, but allow us to characterize the distortion from private information in our setting (Theorem 3).

Delegation Prior to the agent's experimentation decision, the principal can choose the *menu* $D \subseteq \Pi$ of experiments that are available to the agent. This menu can contain any Blackwell experiments that the principal wants, but the principal must allow the agent to refrain from experimentation: D *must* contain the uninformative experiment $(\{0\}, \pi_0)$ with $\pi_0(0|0) = \pi_0(0|1) = 1$.

Formally, a menu D can be written as $D = \{\pi_p\}_{p\in\mathcal{P}} \cup \{\pi_0\}$ for some index set $\mathcal{P}$. Associated with each index $p \in \mathcal{P}$ is a set of types $\Theta_p \subseteq \Theta$ that π_p is targeted at. After observing that the agent selected $\pi_p \in D$, the principal infers that the agent's type is in Θ_p, and therefore updates her belief to an interim belief $\beta_p = E_\sigma[\theta|\theta \in \Theta_p]$.[17] A menu is *implementable* if Θ_p is the set of types that actually select the experiment π_p. We focus on minimal optimal menus without loss of generality; that is, we assume $\Theta_p \neq \emptyset$ for all $p \in \mathcal{P}$.

An appropriately designed menu can allow the principal to perfectly infer the agent's private information from the experiment he chooses. A *screening* menu is a menu D for which $|\Theta_p| = 1$ for all $p \in \mathcal{P}$. For screening menus, it is without loss to associate each index $p \in \mathcal{P}$ with the unique type θ that chooses π_p. Therefore, we can let $\mathcal{P} = \Theta$ and write $D = \{\pi_\theta\}_{\theta\in\Theta} \cup \{\pi_0\}$. Upon learning that the agent has conducted π_θ, the principal first updates her belief to θ before updating again once she observes the experiment's result. The distinguishing feature of screening menus is that, even if $\pi_\theta = \pi_{\theta'} = \pi$ for some $\theta \neq \theta'$, the principal can always determine which type conducted experiment π. In contrast, a *pooling* menu is a menu D for which $|\Theta_p| > 1$ for some $p \in \mathcal{P}$.

Contrary to canonical mechanism design environments, we cannot appeal to the standard revelation principle (Myerson, 1982) to convert any pooling menu into an equivalent screening menu.[18] Knowing which type conducted a certain ex-

[17] In order for the principal's interim beliefs after observing the agent's choice of experiment to be well-defined, we require that each Θ_p is measurable.

[18] We can, however, apply the revelation principle from Doval and Skreta (2022). They show that in a design problem where the designer has limited commitment over how they use an agent's private information (as in our setting), the designer can restrict her attention to mechanisms where each type submits an input message, and the output message is a belief over the agent's type. The designer commits to a Bayes-plausible mapping between input and output messages, and an al-

periment is payoff relevant to both the principal and the agent, since this information changes the distribution over posteriors that an experiment induces.

Example 1. Consider, for instance, a menu $D = \{\pi, \pi_0\}$ consisting only of a single partially informative binary experiment π and the uninformative experiment. If all types conduct experiment π, the principal updates from their prior β_0. Suppose $\beta_0 = 1/3$, $b = 2/3$, $\theta_0 = 1/8$, and $\theta_N = 1/2$. Suppose $S = \{\bar{s}, \underline{s}\}$ where $\pi(\bar{s}|1) = 9/10$ and $\pi(\bar{s}|0) = 1/10$. Then, the principal's posterior after observing $\bar{s}$ is

$$\frac{\pi(\bar{s}|1)\beta_0}{\pi(\bar{s}|1)\beta_0 + \pi(\bar{s}|0)(1-\beta_0)} = \frac{9}{11} > \frac{2}{3}$$

so observing $\bar{s}$ leads the principal to approve the project. Now consider the *screening* menu $D = \{\pi_\theta\}_{\theta \in \Theta} \cup \{\pi_0\}$ where each $\pi_\theta = \pi$. Consider type $\theta = \theta_0$. Knowing that the experiment π comes from type θ_0 induces a posterior

$$\frac{\pi(\bar{s}|1)\theta_0}{\pi(\bar{s}|1)\theta_0 + \pi(\bar{s}|0)(1-\theta_0)} = \frac{9}{16} < \frac{2}{3}$$

and so observing $\bar{s}$ in the screening menu when $\theta = \theta_0$ does not lead the principal to approve the project.

In standard delegation problems, each delegation set D is equivalent to a direct mechanism that maps each agent's type report to their choice from D.[19] Example 1 shows that delegation sets and direct mechanisms are not interchangeable in our setting, because the agent's type report may provide the principal with payoff-relevant information that cannot be inferred from the agent's choice from the delegation set (i.e., with a pooling menu). Consequently, because the principal cannot commit to a decision rule, the direct mechanism associated with a delegation set may not be implementable. In particular, the principal may approve after a positive test result when the agent selects from a delegation set, but may reject after the same result in the associated direct mechanism. Thus, it is not without loss of generality to consider truth-telling direct revelation mechanisms (equivalently, screening menus).

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location rule which maps output messages into actions. In our setting, any delegation set is associated with a canonical mechanism. Screening menus are associated with canonical mechanisms where the mapping between input and output messages is fully revealing; that is, the output message reveals the agent's type.

[19]Truth-telling is automatically optimal for the agent, since the direct mechanism simply assigns them their choice from the mechanism.

Timing To summarize, the timing of the game is as follows:

1. Nature realizes a state $\omega \in \{0, 1\}$.
2. The agent receives a private signal and updates to a belief $\theta \in \Theta$.
3. The principal chooses a menu D.
4. The agent selects an experiment $\pi \in D$, and the principal observes this choice and updates to an interim belief.
5. The result of the experiment is realized, publicly revealed, and the principal updates to a posterior belief.
6. The principal makes an approval decision.

3 Implementability

Before we determine which delegation set maximizes the principal's expected payoff, we first ask which menus are *implementable*. A menu D is implementable if each type finds it optimal to choose the experiment designed for them. That is, if $\theta \in \Theta_p$, an agent of type θ prefers π_p to any other experiment in D. We describe implementability in terms of conditions on the distributions over posteriors an experiment can induce.

Since the agent's cost function is posterior-separable, his payoffs depend on the experiment he selects only through the distribution of posterior beliefs that the experiment induces. However, when the principal cannot infer the agent's private information about the state by observing his choice of experiment, she will have a different posterior than the agent does after observing the experiment's result. Consequently, the same experiment can induce up to three different distributions that are relevant to our analysis: the distribution of the principal's posteriors, the distribution of the agent's posteriors, and the distribution of the principal's posteriors *from the perspective of the agent*.

The first two of these can be described using notation that we have already introduced: if the principal's interim belief after observing the agent's choice is α, then π induces the distribution $\langle \pi | \alpha \rangle$ for her and $\langle \pi | \theta \rangle$ for the agent. Lemma 1 characterizes the last of these distributions in the same terms.[20]

[20]Lemma 1 is complementary to Proposition 1 in Alonso and Câmara (2016), who also study a setting where the priors of the sender and receiver may differ: they compute the receiver's belief as a function of the sender's belief, whereas we instead compute the probability that the sender places

Lemma 1 (Agent's Distribution of Principal's Posteriors). *Suppose α is the principal's interim belief after she observes the agent choose the experiment π. Then the agent places probability $\int_B \left(\frac{\theta}{\alpha}\beta + \frac{1-\theta}{1-\alpha}(1-\beta)\right) d\langle\pi|\alpha\rangle(\beta)$ on the principal updating her belief to $\beta \in B$ after observing the result of π.*

Lemma 1 allows us to describe the set of implementable menus. First, observe that implementable menus must be *individually rational*: they provide each type of agent with an expected payoff of at least zero (their payoff when they choose the totally uninformative experiment π_0). That is, a menu $D = \{\pi_p\}_{p\in\mathcal{P}} \cup \{\pi_0\}$ is individually rational if

$$E_{\langle\pi_p|\beta_p\rangle}\left[\left(\frac{\theta}{\beta_p}\beta + \frac{1-\theta}{1-\beta_p}(1-\beta)\right)U(\beta) - G(\beta|\beta_p)\right] \geq 0 \tag{IRθ-P}$$

holds for all $p \in \mathcal{P}$ and all $\theta \in \Theta_p$. The coefficient on $U(\beta)$ arises because the agent's payoff depends on the *principal*'s belief, rather than his own. Since the principal cannot perfectly infer θ by observing π_p when Θ_p is not a singleton, Lemma 1 shows that from the agent's perspective, the expected value of a function of the principal's beliefs is precisely the expected value of $\left(\frac{\theta}{\beta_0}\beta + \frac{1-\theta}{1-\beta_0}(1-\beta)\right)$ times that function.

In addition to being individually rational, an implementable menu must be *incentive compatible*: if $\theta \in \Theta_p$, then agents of type θ prefer the experiment π_p to any experiment conducted by another type. Formally, a menu $D = \{\pi_p\}_{p\in\mathcal{P}} \cup \{\pi_0\}$ is incentive compatible if

$$\begin{aligned}\theta \in \Theta_p \implies & E_{\langle\pi_p|\beta_p\rangle}\left[\left(\frac{\theta}{\beta_p}\beta + \frac{1-\theta}{1-\beta_p}(1-\beta)\right)U(\beta) - G(\beta|\beta_p)\right] \\ & \geq E_{\langle\pi_{p'}|\beta_{p'}\rangle}\left[\left(\frac{\theta}{\beta_{p'}}\beta + \frac{1-\theta}{1-\beta_{p'}}(1-\beta)\right)U(\beta) - G(\beta|\beta_{p'})\right] \ \forall p' \in \mathcal{P}\end{aligned} \tag{ICθ-P}$$

holds for all $\theta \in \Theta$ and all $p \in \mathcal{P}$. By construction, individual rationality and incentive compatibility are necessary and sufficient conditions for a menu to be implementable.

In principle, pooling can occur in a general manner: that is, the principal could

on the receiver having a certain belief.

try to design a menu where *any* set of types pool together on the same experiment. However, the individual rationality and incentive compatibility conditions place substantial restrictions on the manner in which pooling can occur. If an implementable menu targets the experiment π_p at types $\theta, \theta' \in \Theta_p$ with $\theta' < \theta$, but targets a different experiment $\pi_{p'}$ at some type $\tilde{\theta} \in (\theta', \theta)$, then the agent must be indifferent between π_p and $\pi_{p'}$ regardless of their type. Lemma 2 shows that this allows the principal to construct a better menu $D' = \{\pi_p\}_{p \in \mathcal{P}'} \cup \{\pi_0\}$ by replacing one of these experiments with the other, so that *pooling is local*: That is, for each $p \in \mathcal{P}'$, if $\theta, \theta' \in \Theta_p$ and $\hat{\theta} \in \Theta$, $\theta' \leq \hat{\theta} \leq \theta \Rightarrow \hat{\theta} \in \Theta_p$.

Lemma 2 (Pooling Is Local Without Loss). *For any implementable menu D, there is an implementable menu D' where pooling is local that gives the principal a weakly higher ex ante expected payoff than D.*

A *binary experiment* is (S, π) with $S = \{\underline{s}, \overline{s}\}$ and (without loss) $\pi(\overline{s}|0) \leq \pi(\overline{s}|1)$. If every informative experiment $\pi \not\sim_B \pi_0$ in a menu D is binary, we say that it is a *binary menu*. As observed by Doval and Skreta (2023), we cannot appeal to standard two-state information design arguments to conclude that binary menus are optimal, because the principal faces both incentive compatibility and individual rationality constraints. Nevertheless, we show that binary experiments are without loss in our setting. Intuitively, if D is a menu containing non-binary experiments, replacing each non-binary experiment π_p with the binary experiment π'_p that leads the principal to approve the project with the same probabilities conditional on the state does not lower principal payoffs, but weakens the agents' incentive constraints since $C(\pi_p) > C(\pi'_p)$.

Lemma 3 (Menus Are Binary Without Loss). *For any implementable menu D, there is an implementable binary menu D' that gives the principal a weakly higher ex ante expected payoff than D.*

3.1 Screening vs. Pooling

In classic delegation settings, e.g. Holmström (1977) and Alonso and Matouschek (2008), the principal does not benefit from learning the agent's private information since the principal has delegated the right to make a decision to the agent. Thus, pooling menus are typically optimal.[21] In our setting, since the principal only dele-

[21] The optimal menu in Alonso and Matouschek (2008), for instance, is a pooling menu.

gates the experimentation decision to the agent and retains the right to make an approval decision, learning the agent's private information is beneficial, as the principal can leverage the agent's private information when deciding whether or not to approve the project. This reasoning suggests that a screening menu is optimal.

However, the principal faces a tradeoff between screening and pooling menus. On one hand, the principal benefits directly from screening menus as it allows her to learn the agent's private information. On the other hand, screening makes it more challenging to satisfy lower types' individual rationality constraints (Example 1). Despite this tension, for any pooling menu, the principal can construct a new menu that makes her better off.

Theorem 1 (Screening Dominates Pooling). *If the pooling menu D is implementable, then there exists an implementable screening menu $\tilde{D}$ which gives the principal a weakly higher ex ante expected payoff.*

The intuition is as follows. Given an implementable pooling menu, suppose we construct a screening menu where each type's experiment is a copy of the (without loss, binary) experiment they choose from the pooling menu. This allows the principal to learn θ before he makes an approval decision, while leaving the experiment conducted by each type unchanged. Clearly, this cannot decrease her expected payoff. But it might cause some type θ's participation (IRθ-P) or incentive compatibility (ICθ-P) conditions to fail.

If it does, it must be because learning θ causes the principal to reject the project even after receiving a positive result from the experiment: Otherwise, the experiment gives any type that conducts it the same expected payoff as it did when it was part of the pooling menu.[22] In such cases, we can restore type θ's participation constraint by assigning him an experiment conducted by a lower type, or the uninformative experiment π_0. And as Lemma 10 in the appendix shows, we can restore incentive compatibility by assigning experiments the principal prefers to types above θ. Both of these changes must (weakly) increase the principal's expected payoff even further.

[22] Similarly, any other type's experiment either gives type θ the same payoff (if the principal still approves after a positive result) or a worse payoff (otherwise), so his incentive compatibility constraints are no more stringent.

3.2 Screening Menu Implementability

In light of Theorem 1, we focus on screening menus for the rest of the paper. This simplifies the individual rationality and incentive compatibility constraints facing the principal. Formally, a screening menu $D = \{\pi_\theta\}_{\theta \in \Theta} \cup \{\pi_0\}$ is individually rational if

$$E_{\langle \pi_\theta | \theta \rangle} \left[U(\beta) - G(\beta|\theta) \right] \geq 0 \tag{IRθ}$$

holds for all θ, and is incentive compatible if

$$E_{\langle \pi_\theta | \theta \rangle} \left[U(\beta) - G(\beta|\theta) \right] \geq E_{\langle \pi_{\theta'} | \theta' \rangle} \left[\left(\tfrac{\theta}{\theta'} \beta + \tfrac{1-\theta}{1-\theta'} (1-\beta) \right) U(\beta) - G(\beta|\theta') \right] \tag{ICθ}$$

holds for all $\theta, \theta' \in \Theta$.

Even though our setting does not involve transfers and the principal cannot incentivize the agent by altering his approval threshold, we can still offer a characterization of the class of implementable menus that is reminiscent of standard transferable utility contracting results (e.g., Maskin and Riley (1984)).

Proposition 1 (Implementable Menus). *Let $D = \{\pi_\theta\}_{\theta \in \Theta} \cup \{\pi_0\}$ be a screening menu.*

i. If D satisfies

(EC) *Envelope Condition: For all θ,*

$$E_{\langle \pi_\theta | \theta \rangle}[U(\beta)] - C(\pi_\theta) = \sum_{\theta_i < \theta} (\theta_{i+1} - \theta_i) E_{\langle \pi_{\theta_i} | \theta_i \rangle} \left[\left(\tfrac{\beta - \theta_i}{\theta_i (1-\theta_i)} \right) U(\beta) \right]; \tag{ECθ}$$

(M) *Monotonicity: For all $\theta \geq \theta'$,*

$$E_{\langle \pi_\theta | \theta \rangle} \left[\left(\frac{\beta - \theta}{\theta(1-\theta)} \right) U(\beta) \right] \geq E_{\langle \pi_{\theta'} | \theta' \rangle} \left[\left(\frac{\beta - \theta'}{\theta'(1-\theta')} \right) U(\beta) \right], \tag{M(θ,θ')}$$

then D is implementable.

ii. If D is implementable, it satisfies (M) (monotonicity).

iii. If D is implementable, there exists an implementable screening menu $D' = \{\pi'_\theta\}_{\theta \in \Theta} \cup \{\pi_0\}$ that satisfies (EC) such that $E_{\langle \pi'_\theta | \theta \rangle}[W(\beta)] \geq E_{\langle \pi_\theta | \theta \rangle}[W(\beta)]$ for each $\theta \in \Theta$.

The key insight behind this characterization is that the law of iterated expectations allows us to decompose the expectations in (ICθ) into expectations of U *conditional on the state*. This decomposition has two implications. First, because experiments are multidimensional objects, these conditional expectations can vary separately, just like transfers and quantities in, e.g., Maskin and Riley (1984) and Mussa and Rosen (1978). And since types differ precisely in the probabilities that they assign to each state, they can be screened by offering a lower type an experiment that yields higher expected utility conditional on state 0 and by offering a higher type an experiment that yields higher expected utility conditional on state 1. Formally, every implementable screening menu must satisfy the following condition: if $\theta > \theta'$, then

$$E_{\langle\pi_\theta|\theta\rangle}[U(\beta)|\omega = 1] - C(\pi_\theta) \geq E_{\langle\pi_{\theta'}|\theta'\rangle}[U(\beta)|\omega = 1] - C(\pi_{\theta'});$$
$$E_{\langle\pi_\theta|\theta\rangle}[U(\beta)|\omega = 0] - C(\pi_\theta) \leq E_{\langle\pi_{\theta'}|\theta'\rangle}[U(\beta)|\omega = 0] - C(\pi_{\theta'}).$$

As in the standard transferable utility model, this screening is most effective when local incentive compatibility constraints bind. However, these local constraints — captured in (ECθ) — are not enough to ensure that a menu is incentive compatible. Instead, ensuring that global deviations in type reports are not advantageous requires an additional monotonicity condition. Intuitively, a monotonic menu is one in which the difference between the conditional expected payoff in state $\omega = 1$ and state $\omega = 0$ is increasing in the type report.

Second, the individual rationality constraint for any type $\theta > \theta_0$ is redundant, just like in a transferable utility setting. Since U is increasing, its expectation conditional on $\omega = 1$ is always higher than its expectation conditional on $\omega = 0$, and so the expectations on the right hand side of (ICθ) are higher for $\theta > \theta_0$ than for $\theta = \theta_0$.

3.3 Discussion

A common measure of the performance of a diagnostic test in the medical literature is *Youden's index* (Youden, 1950). For a binary test, Youden's index is the difference between the true positive rate and the false positive rate. In our notation, we denote the Youden's index for a binary experiment π as $\eta(\pi) := \pi(\bar{s}|1) - \pi(\bar{s}|0)$.

In the literature, tests with higher Youden's indices are often considered to be better performing. This is not entirely congruent with Blackwell's Theorem: If one binary experiment is Blackwell-more informative than another, it must have a higher Youden's index, but the converse is not true. However, a clarification reveals the relevance of the Youden's index to implementation: tests with higher Youden's indices are better performing *for* an agent with transparent motives *when* the hypothesis being tested is more likely to be true.

Observe that when a type-θ agent chooses a binary experiment π_θ from a screening menu, his expected payoff can be written

$$E_{\langle \pi_\theta | \theta \rangle}[U(\beta)] - C(\pi_\theta) = (\theta \eta(\pi_\theta) + \pi_\theta(\bar{s}|0))u - C(\pi_\theta). \tag{1}$$

Thus, since the cost of experimentation C is prior-independent, an experiment's Youden's index is a sufficient statistic for the way that the agent's payoff from conducting it depends on her type. In particular, if the Youden's index of one experiment is higher than that of another, then whenever some type chooses the former over the latter, all higher types do as well. That is, *the Youden's index is the aspect of the experiment in which the agent's payoff has the single crossing property in θ.* It should not be surprising, then, that the monotonicity constraint (M) is equivalent to requiring $\eta(\pi_\theta)$ to be non-decreasing in θ. Likewise, the envelope condition (EC) means that the menu gives each type a payoff equal to the weighted sum of the lower types' Youden indices.

Proposition 2 (Youden's Index and Implementation). *Suppose that $D = \{\pi_\theta\}_{\theta \in \Theta} \cup \{\pi_0\}$ is a binary screening menu that is individually rational.*

i. D satisfies the monotonicity condition (M) if and only if $\eta(\pi_\theta)$ is non-decreasing in θ.

ii. D satisfies the envelope condition (EC) if and only if $E_{\langle \pi_\theta | \theta \rangle}[U(\beta)] - C(\pi_\theta) = u \sum_{\theta_i < \theta} (\theta_{i+1} - \theta_i) \eta(\pi_{\theta_i})$ for all $\theta \in \Theta$.

4 Optimal Delegation

In Section 3, we showed that screening menus are without loss of generality for the principal. Now, we study the principal's problem of designing such a menu. First, in Section 4.1, we show that an optimal menu exists, and characterize it. Sections 4.2 and 4.3 then show how asymmetric information makes the optimal menu inefficient.

4.1 The Principal's Problem

Since the optimal menu from the principal's perspective is a screening menu, Proposition 1 allows us to write the principal's problem as

$$\max_{\{\pi_\theta\}_{\theta\in\Theta}\in\Pi^\Theta} \sum_{\theta\in\Theta} E_{\langle\pi_\theta|\theta\rangle}[W(\beta)]\sigma(\theta) \tag{COPT}$$

$$\text{s.t.} \quad \begin{array}{l} (EC): \; E_{\langle\pi_{\theta_n}|\theta_n\rangle}[U(\beta)-G(\beta|\theta_n)] = \sum_{i<n}(\theta_{i+1}-\theta_i)E_{\langle\pi_{\theta_i}|\theta_i\rangle}\left[\left(\frac{\beta-\theta_i}{\theta_i(1-\theta_i)}\right)U(\beta)\right] \;\forall n; \\ (M): \; E_{\langle\pi_{\theta_n}|\theta_n\rangle}\left[\left(\frac{\beta-\theta_n}{\theta_n(1-\theta_n)}\right)U(\beta)\right] \geq E_{\langle\pi_{\theta_{n-1}}|\theta_{n-1}\rangle}\left[\left(\frac{\beta-\theta_{n-1}}{\theta_{n-1}(1-\theta_{n-1})}\right)U(\beta)\right] \;\forall n>0. \end{array}$$

Theorem 2 extends the duality results of Doval and Skreta (2023), allowing us to write (COPT) as an unconstrained problem using the method of Lagrange.[23] In particular, we extend the key result in Doval and Skreta (2023) to solve problems where the choice variable is a menu of experiments and to allow for constraint functions that are not upper semi-continuous. As we show, the Lagrangian (COPT′) has a solution, and so (COPT) must as well.

Theorem 2 (Existence and Uniqueness). *A solution exists to* (COPT)*. This solution is unique (up to Blackwell equivalence) and binary. Furthermore, there exists a family of non-negative Lagrange multipliers* $\{\lambda_n^*\}_{n=0}^N, \{\delta_n^*\}_{n=1}^N$ *such that* $D^* = \{\pi_\theta^*\}_{\theta\in\Theta}\cup\{\pi_0\}$ *solves* (COPT) *if and only if* $\{\pi_{\theta_n}^*\}_{n=0}^N$ *solves*

$$\begin{aligned} \max_{\{\pi_{\theta_n}\}_{n=0}^N\in\Pi^{N+1}} \sum_{n=0}^N & E_{\langle\pi_{\theta_n}|\theta_n\rangle}[W(\beta)]\sigma(\theta_n) \\ & + \lambda_n^*\left(E_{\langle\pi_{\theta_n}|\theta_n\rangle}[U(\beta)-G(\beta|\theta_n)] - \sum_{i<n}(\theta_{i+1}-\theta_i)E_{\langle\pi_{\theta_i}|\theta_i\rangle}\left[\left(\frac{\beta-\theta_i}{\theta_i(1-\theta_i)}\right)U(\beta)\right]\right) \\ & + \delta_n^*\left(E_{\langle\pi_{\theta_n}|\theta_n\rangle}\left[\left(\frac{\beta-\theta_n}{\theta_n(1-\theta_n)}\right)U(\beta)\right] - E_{\langle\pi_{\theta_{n-1}}|\theta_{n-1}\rangle}\left[\left(\frac{\beta-\theta_{n-1}}{\theta_{n-1}(1-\theta_{n-1})}\right)U(\beta)\right]\right)\mathbf{1}_{n>0}. \end{aligned} \tag{COPT′}$$

Furthermore, for all $n>0$,

$$\delta_n^*\left(E_{\langle\pi_{\theta_n}^*|\theta_n\rangle}\left[\left(\frac{\beta-\theta_n}{\theta_n(1-\theta_n)}\right)U(\beta)\right] - E_{\langle\pi_{\theta_{n-1}}^*|\theta_{n-1}\rangle}\left[\left(\frac{\beta-\theta_{n-1}}{\theta_{n-1}(1-\theta_{n-1})}\right)U(\beta)\right]\right) = 0. \tag{CS}$$

Moreover, the maximized values of (COPT) *and* (COPT′) *are equal.*

By rearranging the terms in (COPT′), we can write the unique solution $D^* =$

[23] Other authors have employed the Lagrangian approach to solve delegation problems. See, for instance, Amador and Bagwell (2013), Guo (2016), and Kartik et al. (2021).

$\{\pi^*_\theta\}_{\theta\in\Theta} \cup \{\pi_0\}$ to (COPT) as the tuple of (unique) solutions to a set of type-by-type problems: For each n,

$$\pi^*_{\theta_n} \in \arg\max_{\pi\in\Pi} E_{\langle\pi|\theta_n\rangle}\left[\underbrace{W(\beta)\sigma(\theta_n) + \lambda^*_n(U(\beta) - G(\beta|\theta_n))}_{\text{Surplus Term } S(\beta,\theta_n)} - \underbrace{\rho^*_n\left(\frac{\beta-\theta_n}{\theta_n(1-\theta_n)}\right)U(\beta)}_{\text{Rent Term } R(\beta,\theta_n)}\right],$$

$$\text{where } \rho^*_n := \left(\delta^*_{n+1} + (\theta_{n+1}-\theta_n)\sum_{i=n+1}^{N}\lambda^*_i\right)\mathbf{1}_{n<N} - \delta^*_n\mathbf{1}_{n>0}. \qquad (\text{TBT}\theta_n)$$

These solutions thus maximize virtual surplus: that is, a weighted sum of the principal and agent's expected payoffs (the *surplus term* $E_{\langle\pi|\theta_n\rangle}[S(\beta,\theta_n)]$), minus a *rent term* $E_{\langle\pi|\theta_n\rangle}[R(\beta,\theta_n)]$ (for a binary experiment, a scalar multiple of the Youden's index discussed in Section 3.3). Figure 1 depicts the objective function of these type-by-type problems. As we show in Section 4.3, the latter term distorts the problem's solution away from efficiency, except in the case of the highest type θ_N.

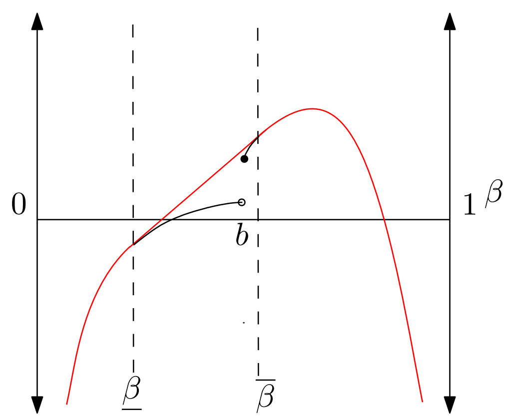

Figure 1: Solution to the Type-by-Type Problems. The type-by-type objective functions (black) are each strictly concave in β on $[0,b)$ and on $[b,1]$, with a jump discontinuity at b. This gives each of the problems a unique binary solution (Lemma 11 in the appendix), which we use to show uniqueness in Theorem 2.

Exclusion

The principal may find it optimal to *exclude* some types by not offering them an informative experiment that they are willing to conduct; i.e., we may have $\pi^*_\theta \sim_B \pi_0$ for some types. Lemma 4 shows that the set of *included types* $\tilde{\Theta} := \{\theta \in \Theta : \pi^*_\theta \not\sim_B \pi_0\}$ forms an upper set: that is, if it is optimal for the principal to exclude one type θ, it must also be optimal for him to exclude any type $\theta' \leq \theta$ with worse news about the state of the world.

Lemma 4 (Exclusion Is At The Low End). $\tilde{\Theta} = \{\theta \in \Theta : \theta \geq \underline{\theta}\}$ *for some* $\underline{\theta} \in \Theta$*.*

Intuitively, a binary experiment π is uninformative if and only if its Youden's index $\eta(\pi) = \pi(\bar{s}|1) - \pi(\bar{s}|0)$ is zero. Since (M) is equivalent to the monotonicity of the Youden's index, it follows that any implementable menu of binary experiments — including D^* — must exclude only at the low end of the distribution of private information.

The Monotonicity Constraint

By its nature, the monotonicity constraint (M) must bind on *intervals*: that is, if (M(θ'', θ')) binds for some $\theta'' > \theta'$, (M(θ, θ')) must also bind for all $\theta \in (\theta', \theta'')$. But unlike in standard contracting problems where the choice variable is one-dimensional, this does not guarantee that each type in such an interval conducts the same experiment — just that those experiments have the same Youden's index η. However, Lemma 5 shows that in the *optimal* menu D^*, if two types conduct experiments with the same Youden's index, the experiments must be identical.[24]

Lemma 5 (Experiments Where Monotonicity Binds). *Suppose* $(M(\theta'', \theta'))$ *binds for some* $\theta'' > \theta'$*. Then for each* $\theta \in [\theta', \theta'']$*,* $\pi^*_\theta = \pi^*_{\theta''}$*.*

The intuition is as follows. Suppose monotonicity binds on the interval I, so that the experiments π^*_θ conducted by types $\theta \in I$ have identical Youden's indices η. Then on I, both sides of the envelope condition (ECθ) are linear in θ, with the same coefficient η. It follows that (ECθ) can only be satisfied on I if every experiment π^*_θ conducted by a type $\theta \in I$ has the same cost $C(\pi_\theta)$. If multiple experiments have the same Youden's index and the same cost, the principal optimally includes in D^* the one with the lowest false positive rate.

4.2 The Pareto Frontier

To see how experimentation is distorted by the need to induce the agent to voluntarily reveal his private information, we must first characterize what it is distorted away *from*. Hence, for each type of agent, we describe the set of experiments

[24] More specifically, they must be identical save for the labeling that identifies the type that conducted them.

that are Pareto efficient, given the agent's type. Since transfers are not possible in our setting, these sets are not singletons. Instead, there is a *Pareto frontier* of efficient experiments for each type, with one end of that frontier being optimal for the principal, and the other being optimal for the agent.

Formally, an experiment π is *Pareto efficient for type* θ if there is no π' such that $E_{\langle \pi'|\theta\rangle}[W(\beta)] \geq E_{\langle \pi|\theta\rangle}[W(\beta)]$ and $E_{\langle \pi'|\theta\rangle}[U(\beta)] - C(\pi') \geq E_{\langle \pi|\theta\rangle}[U(\beta)] - C(\pi)$, with one of the inequalities strict. As one might expect, such experiments are precisely those that solve a social planner's problem.

Proposition 3 (Efficiency and the Social Planner's Problem). *π is Pareto efficient for type θ if and only if there exist $\lambda_p \geq 0$ and $\lambda_a \geq 0$, with one inequality strict, such that*

$$\langle \pi|\theta\rangle \in \arg \max_{\tau \in \Delta(\Delta(\Omega))} \left\{ E_\tau \left[\lambda_p W(\beta) + \lambda_a (U(\beta) - G(\beta|\theta)) \right] \; s.t. \; E_\tau \beta = \theta \right\}. \quad \text{(SPP}\theta\text{)}$$

Together, Proposition 3 and Lemma 11 allow us to characterize the type-θ Pareto frontier geometrically. In particular, Lemma 11 shows that when $\lambda_a \neq 0$, (SPPθ) has a unique solution characterized by one of three alternatives. First, if there are posteriors $\underline{\beta}$ and $\overline{\beta}$ such that $\underline{\beta} < \theta < b < \overline{\beta}$ and

$$-\lambda_a G'(\underline{\beta}|\theta) = \lambda_p(w_1 - w_0) - \lambda_a G'(\overline{\beta}|\theta) = \frac{\lambda_p(w_0 + (w_1 - w_0)\overline{\beta}) + \lambda_a(u - G(\overline{\beta}|\theta)) + \lambda_a G(\underline{\beta}|\theta)}{\overline{\beta} - \underline{\beta}},$$

or equivalently, since $b \equiv \frac{-w_0}{w_1 - w_0}$,

$$\begin{aligned} \frac{\lambda_p(-w_0)}{\lambda_a b} &= G'(\overline{\beta}|\theta) - G'(\underline{\beta}|\theta) \text{ and} \\ 0 &= \underbrace{u - G(\overline{\beta}|\theta) - G'(\overline{\beta}|\theta)(b - \overline{\beta})}_{\text{tangent line to } U - G(\cdot|\theta) \text{ at } \overline{\beta}} + \underbrace{G(\underline{\beta}|\theta) + G'(\underline{\beta}|\theta)(b - \underline{\beta})}_{\text{tangent line to } U - G(\cdot|\theta) \text{ at } \underline{\beta}}, \end{aligned} \tag{2}$$

then (SPPθ)'s unique solution is given by the unique Bayes-plausible distribution that induces the posteriors $\underline{\beta}$ and $\overline{\beta}$. Geometrically, this condition means that the tangent lines to the agent's value function at $\underline{\beta}$ and $\overline{\beta}$ must cross at b, and the difference in their slopes (and since $G(\cdot|\theta)$ is strictly convex, the distance $\overline{\beta} - \underline{\beta}$) is increasing in the relative Pareto weight on the principal.[25] Figure 2 illustrates.

[25] To understand why, recall that W is continuous, affine to the right of the approval threshold b, and zero to the left of b. Then adding $\lambda_p/\lambda_a W(\beta)$ to the agent's value function (thus yielding the objective in (SPPθ)) essentially rotates the function — and thus its tangent lines — upward to the right of b. Consequently, the tangent lines to the value function in (SPPθ) at $\underline{\beta}$ and $\overline{\beta}$ *coincide* (as

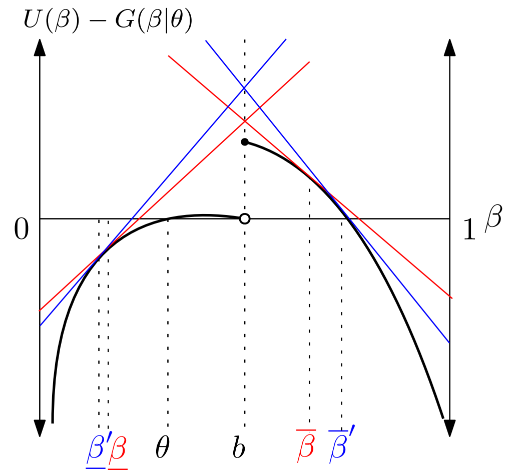

Figure 2: Characterization of the Pareto Frontier. Each experiment that is efficient for type θ induces a pair of posteriors such that the tangent lines to the agent's value function at those posteriors cross at b. When the Pareto weight on the principal is higher relative to the weight on the agent, those posteriors are further apart, and the tangent lines cross at a higher point: The experiment that induces $\underline{\beta}'$ and $\overline{\beta}'$ solves (SPPθ) for a higher value of λ_p/λ_a, while the experiment that induces $\underline{\beta}$ and $\overline{\beta}$ solves (SPPθ) for a lower value of λ_p/λ_a.

If the Pareto weight assigned to the principal is small enough relative to the Pareto weight on the agent, (2) becomes impossible to satisfy. Then, Lemma 11's second alternative[26] says that (2)'s unique solution is the unique Bayes-plausible distribution that induces the posterior beliefs $\underline{\beta}$ and b, where $\underline{\beta} < \theta$ is pinned down by

$$-G(\underline{\beta}|\theta) - G'(\underline{\beta}|\theta)(b - \underline{\beta}) = u - G(b|\theta), \tag{3}$$

Geometrically, this requires that at the approval threshold b, the agent's value function coincides with the tangent line to his value function at $\underline{\beta}$. Thus, in this case, (SPPθ)'s solution is precisely the distribution of posteriors that maximizes the agent's payoff when his type is common knowledge.

Corollary 1 summarizes.

Corollary 1. *π is Pareto efficient for type θ if and only if one of the following holds:*

is necessary for the two posteriors to support a solution) exactly when tangent lines to the agent's value function $U - G(\cdot|\theta)$ *cross* at the approval threshold, and have slopes that differ by the relative weight λ_p/λ_a that the planner places on the principal times the slope of his value function.

[26]Lemma 11's third alternative is ruled out: By assumption, there is an informative experiment that gives the agent a nonnegative payoff — and thus Pareto improves upon the uninformative experiment π_0.

i. $\text{supp}\langle\pi|\theta\rangle = \{\underline{\beta}, \overline{\beta}\}$, $\underline{\beta} < \theta < b < \overline{\beta}$, *and* (2) *holds for some* $\lambda_a, \lambda_p > 0$.

ii. $\text{supp}\langle\pi|\theta\rangle = \{\underline{\beta}, b\}$, $\underline{\beta} < \theta$, *and* (3) *holds.*

iii. π *is fully informative:* $\pi \sim_B \pi_\infty$, *where* π_∞ *is the binary experiment with* $\pi_\infty(\overline{s}|1) = \pi(\underline{s}|0) = 1$.

4.3 Distortion

Corollary 1 describes the experiments that are *on* each type's Pareto frontier. This allows us to describe how and why that type's experiment from the principal's optimal menu is distorted *away* from the frontier. In the case of the highest type, the answer is simple: it isn't. Thus, we arrive at the classical result that there is *no distortion at the top*.

Proposition 4 (No Distortion at the Top). *In the principal's optimal screening menu, the high type's experiment* $\pi^*_{\theta_N}$ *is Pareto efficient for type* θ_N.

The intuition is standard when the highest type's monotonicity constraint $M(\theta_N, \theta_{N-1})$ does not bind: Then, the Lagrange multiplier δ^*_N is zero, and so the type-by-type problem for the highest type coincides with the social planner's objective in (SPPθ), given appropriate Pareto weights.[27] If, on the other hand, the highest type's monotonicity constraint binds, the argument is more subtle: If $\pi^*_{\theta_N}$ was not on the Pareto frontier, then the principal can construct a strictly better implementable menu by replacing $\pi^*_{\theta_N}$ with some experiment π' for all types that conduct $\pi^*_{\theta_N}$. This can be done in a way that makes the principal strictly better off without affecting agent payoffs since $\pi^*_{\theta_N}$ is not on the Pareto frontier.

Just like in optimal contracting problems with transfers, however, there *is* distortion for types other than the highest. When a type-θ agent chooses the experiment $\pi_{\theta'}$ designed for some other type $\theta' \neq \theta$, the principal shifts his interim belief to θ', rather than θ. Because of this, the type-θ incentive compatibility constraint — and hence the Lagrangian term in (COPT′) — contains an extra coefficient on $U(\beta)$ that depends on the posterior belief and is not present in the social planner's problem for *any* Pareto weights λ_a and λ_p. This distorts the tangent line conditions in (2), and thus the experiment they pin down. In particular, when the principal does

[27] Specifically, for Pareto weights $\lambda_p = \sigma(\theta)$ and $\lambda_a = \lambda^*_\theta$.

not exclude θ_n — i.e., when $\pi^*_{\theta_n}$ is not the uninformative experiment π_0 — Lemma 11 shows that $\langle \pi^*_{\theta_n} | \theta_n \rangle$'s support $\{\underline{\beta}, \overline{\beta}\}$ is characterized by

$$-\lambda^*_n G'(\underline{\beta}|\theta_n) = \frac{\sigma(\theta_n)W(\overline{\beta}) + \lambda^*_n u - R(\overline{\beta}, \theta_n) - \lambda^*_n (G(\overline{\beta}|\theta_n) - G(\underline{\beta}|\theta_n))}{\overline{\beta} - \underline{\beta}}, \quad (4)$$

and when $\overline{\beta} > b$,

$$-\lambda^*_n G'(\underline{\beta}|\theta_n) = \sigma(\theta_n)(w_1 - w_0) + R_\beta(\overline{\beta}, \theta_n) - \lambda^*_n G'(\overline{\beta}|\theta_n). \quad (5)$$

Equivalently, since U and $R(\cdot, \theta_n)$ are affine on $[b, 1]$, we have[28]

$$\underbrace{-G(\underline{\beta}|\theta_n) - G'(\underline{\beta}|\theta_n)(b - \underline{\beta})}_{\text{tangent line to } U - G(\cdot|\theta_n) \text{ at } \underline{\beta}} = \underbrace{u - G(\overline{\beta}|\theta_n) - G'(\overline{\beta}|\theta_n)(b - \overline{\beta})}_{\text{tangent line to } U - G(\cdot|\theta_n) \text{ at } \overline{\beta}} - \underbrace{R(b, \theta_n)/\lambda^*_n}_{\text{distortion term}}; \quad (6)$$

and when $\overline{\beta} > b$, $G'(\overline{\beta}|\theta_n) - G'(\underline{\beta}|\theta_n) = \frac{\sigma(\theta_n)(w_1 - w_0) + R_\beta(\overline{\beta}, \theta_n)}{\lambda^*_n}$.

Observe that there is an additional "distortion term" in (6) that is not present in (2). This ensures that instead of being characterized by tangent lines that *cross* at the approval threshold, the posteriors $\underline{\beta}$ and $\overline{\beta}$ that support $\langle \pi^*_\theta | \theta_n \rangle$ are characterized by tangent lines that *have a gap equal to the distortion term*. When type θ_n's downward monotonicity constraint $M(\theta_n, \theta_{n-1})$ does not bind, we can show that the distortion term is positive (Lemma 17 in the appendix); consequently, $\underline{\beta}$ is inefficiently high given $\overline{\beta}$, and $\overline{\beta}$ is inefficiently high given $\underline{\beta}$. In fact, this is also true for types whose downward monotonicity constraint *does* bind, since their experiment from the optimal menu is the same as the highest type below them for whom such constraints do not bind (Lemma 5).[29] Figure 3 illustrates.

[28]When $\overline{\beta} = b$, (6) follows immediately from (4); when $\overline{\beta} > b$, application of (5) yields

$$\begin{aligned} -G(\underline{\beta}|\theta_n) - G'(\underline{\beta}|\theta_n)(b - \underline{\beta}) &= -G(\overline{\beta}|\theta_n) - G'(\overline{\beta}|\theta_n)(b - \overline{\beta}) + u + \frac{1}{\lambda^*_n}\begin{pmatrix} \sigma(\theta_n)W(\overline{\beta}) - R(\overline{\beta}, \theta_n) \\ -\left(\sigma(\theta_n)(w_1 - w_0) - R_\beta(\overline{\beta}, \theta_n)\right)(\overline{\beta} - b) \end{pmatrix} \\ &= -G(\overline{\beta}|\theta_n) - G'(\overline{\beta}|\theta_n)(b - \overline{\beta}) - R(b, \theta_n)/\lambda^*_n. \end{aligned}$$

[29]In Theorem 3 (i), we show something stronger: whether or not a type's downward monotonicity constraint binds, the principal's optimal menu yields an experiment with an inefficiently low false negative rate.

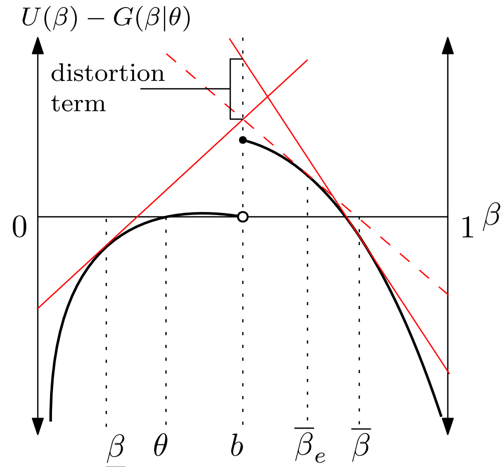

Figure 3: Distortion of the type-θ experiment. Suppose a binary experiment π_θ induces the posterior $\underline{\beta}$. For π^*_θ to be efficient for type θ, it would need to also induce $\overline{\beta}_e$: At the approval threshold b, the tangent line to the agent's value function at $\overline{\beta}_e$ crosses the tangent line to the value function at $\underline{\beta}$. But if π^*_θ solves the principal's type-by-type problem (TBTθ_n), it must induce $\overline{\beta}$ instead of $\overline{\beta}_e$, since the tangent lines to the value function at $\overline{\beta}_e$ and $\underline{\beta}$ differ by the distortion term at β.

This geometric characterization captures the key way that the type-θ agent's experiment π^*_θ from the optimal menu differs from *every* experiment that is Pareto efficient for him. But not all of these differences can be accurately described as *distortion*: Since there are no transfers, the fact that an experiment is efficient does not imply that it makes *both* the principal and the agent better off than π^*_θ. Hence, we focus on comparing the experiments from the optimal menu to *Pareto improving* experiments on the Pareto frontier.

Theorem 3 shows that when faced with the optimal screening menu, every included type other than the highest θ_N and (possibly) the lowest $\underline{\theta}$ chooses an experiment whose false negative rate is too high. Intuitively, since each posterior induced by the optimal experiment is inefficiently high given the other, one can show that if π_{θ_n} Pareto improves upon $\pi^*_{\theta_n}$, it either (a) is Blackwell-more informative than $\pi^*_{\theta_n}$ or (b) corresponds to a *rightward shift* of the posteriors induced by $\pi^*_{\theta_n}$. In either case, it must have a lower false negative rate than $\pi^*_{\theta_n}$. Thus, private information results in an inefficiently low chance of approval in the state where the project is worth approving.

Theorem 3 (Distortion Everywhere Else). *Let $\{\pi^*_\theta\}_{\theta\in\Theta}$ be the principal's optimal menu.*

i. For each included type $\theta \in \tilde{\Theta}$ lower than the highest θ_N, π^*_θ is not Pareto efficient for type θ.

ii. For each included type $\theta \in \tilde{\Theta}$ other than the highest θ_N and the lowest $\underline{\theta}$, π^*_θ has an inefficiently high false negative rate: For any π_θ on the Pareto frontier for type θ that Pareto-improves upon π^*_θ for type θ,

$$\pi_\theta(\underline{s}|1) < \pi^*_\theta(\underline{s}|1).$$

iii. Exclusion is inefficient: For every excluded type, there is a Pareto efficient experiment that Pareto improves upon the totally uninformative experiment π_0.

Theorem 3 characterizes how π^*_θ is distorted relative to any Pareto improvement which is on the Pareto frontier. It turns out that it differs from the principal's take-it-or-leave-it, complete information offer π^P_θ in the same way, even though that experiment is only a Pareto improvement on π^*_θ for the lowest included type $\underline{\theta}$. In fact, Proposition 5 shows a slightly stronger result: any experiment that is better than π^*_θ for the principal, satisfies the agent's participation constraint, and is on the Pareto frontier has a lower false negative rate than π^*_θ.

Proposition 5 (Optimal Menu vs Complete Information Benchmark). *For any π_θ on the Pareto frontier for type $\underline{\theta} < \theta < \theta_N$ such that $E_{\langle \pi_\theta|\theta\rangle}[W(\beta)] \geq E_{\langle \pi^*_\theta|\theta\rangle}[W(\beta)]$ and $E_{\langle \pi_\theta|\theta\rangle}[U(\beta) - G(\beta|\theta)] \geq 0$,*

$$\pi_\theta(\underline{s}|1) < \pi^*_\theta(\underline{s}|1).$$

5 Conclusion

This paper studies the optimal delegation of experiments to privately informed agents. Based on the characterization results, we offer three main conclusions.

First, screening menus can screen agent types by offering different payoffs in each state of the world. In our drug approval application, pharmaceutical companies with better news select clinical trials from screening menus that offer a higher expected payoff conditional on the drug being effective than their lower confidence counterparts. That is, the difference between the true and false positive rate (the Youden's index) must be increasing in the agent's type report.

Second, despite the fact that participation constraints are more difficult to satisfy in screening menus, we find that screening dominates pooling. This contrasts with the standard results in the delegation literature, since, in our setting, the principal retains the right to make an approval decision. By delegating information acquisition to a privately informed agent through a screening menu, the principal learns the agent's private information and can leverage this private information when making an approval decision.

Finally, in the optimal menu there is no distortion for the highest type. However, there is distortion for all other types. Lower types are either excluded, which is inefficient, or conduct an experiment with an inefficiently high false negative rate. This results in an inefficiently low approval rate when the state of the world is high. In our drug regulation application, too many good drugs are rejected.

Appendix

Corollary 2. *If D is implementable, there exists an implementable menu $D' = \{\pi_p\}_{p \in \mathcal{P}'} \cup \{\pi_0\}$ and $\{a_k\}_0^K$ such that $a_k > a_{k-1}$ for each k, and for each $p \in \mathcal{P}$, $\Theta_p = (a_{k-1}, a_k] \cap \Theta$ for some k. Hence, it is without loss of generality to let $\mathcal{P} = \{1, \dots, K\}$.*

Proof. Follows immediately from Lemma 2. □

Lemma 6 (Implementability). *Let $D = \{\pi_p\}_{p \in \mathcal{P}} \cup \{\pi_0\}$ be a menu. Define the mapping $p^* : \Theta \to \mathcal{P}$ by $\theta \mapsto p^*(\theta)$ where $\theta \in \Theta_{p^*(\theta)}$.*

i. If D satisfies

(EC-P) *Envelope Condition: For all θ,*

$$E_{\langle \pi_{p^*(\theta)} | \beta_{p^*(\theta)} \rangle} \left[\left(\tfrac{\theta}{\beta_{p^*(\theta)}} \beta + \tfrac{1-\theta}{1-\beta_{p^*(\theta)}} (1-\beta) \right) U(\beta) \right] - C(\pi_{p^*(\theta)}) = \sum_{\theta_i < \theta} (\theta_{i+1} - \theta_i) E_{\langle \pi_{p^*(\theta_i)} | \beta_{p^*(\theta_i)} \rangle} \left[\left(\tfrac{\beta - \beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})} \right) U(\beta) \right]; \quad \text{(EC}\theta\text{-P)}$$

(M-P) *Monotonicity: For all $\theta \geq \theta'$,*

$$E_{\langle \pi_{p^*(\theta)} | \beta_{p^*(\theta)} \rangle} \left[\left(\tfrac{\beta - \beta_{p^*(\theta)}}{\beta_{p^*(\theta)}(1-\beta_{p^*(\theta)})} \right) U(\beta) \right] \geq E_{\langle \pi_{p^*(\theta')} | \beta_{p^*(\theta')} \rangle} \left[\left(\tfrac{\beta - \beta_{p^*(\theta')}}{\beta_{p^*(\theta')}(1-\beta_{p^*(\theta')})} \right) U(\beta) \right],$$

then D is implementable.

ii. *If D is implementable, it satisfies (M-P).*

iii. *If D is implementable, there exists an implementable menu* $\hat{D} = \{\hat{\pi}_p\}_{p\in\mathcal{P}} \cup \{\pi_0\}$ *that satisfies (EC-P) such that* $E_{\langle\hat{\pi}_p|\beta_p\rangle}[W(\beta)] \geq E_{\langle\pi_p|\beta_p\rangle}[W(\beta)]$ *for each* $p \in \mathcal{P}$.

Proof. Define the type-θ agent's value from choosing $\pi'_{p^*(\theta')}$ from a menu $D' = \{\pi'_p\}_{p\in\mathcal{P}} \cup \{\pi_0\}$ as

$$V(\theta,\theta'|D') = E_{\langle\pi'_{p^*(\theta')}|\beta_{p^*(\theta')}\rangle}\left[\left(\frac{\theta}{\beta_{p^*(\theta')}}\beta + \frac{1-\theta}{1-\beta_{p^*(\theta')}}(1-\beta)\right)U(\beta)\right] - C(\pi'_{p^*(\theta')}).$$

Observe that

$$\begin{aligned} V(\theta,\theta'|D') &= V(\theta',\theta'|D') + (\theta-\theta')E_{\langle\pi'_{p^*(\theta')}|\beta_{p^*(\theta')}\rangle}\left[\left(\tfrac{\beta}{\beta_{p^*(\theta')}} + \tfrac{1-\beta}{1-\beta_{p^*(\theta')}}\right)U(\beta)\right] \\ &= V(\theta',\theta'|D') + (\theta-\theta')E_{\langle\pi'_{p^*(\theta')}|\beta_{p^*(\theta')}\rangle}\left[\left(\tfrac{\beta-\beta_{p^*(\theta')}}{\beta_{p^*(\theta')}(1-\beta_{p^*(\theta')})}\right)U(\beta)\right]. \quad (7) \end{aligned}$$

(i): Suppose D satisfies (EC-P) and (M-P).

By Bayes' rule, for each $p \in \mathcal{P}$, $\beta_p = \sum_{\theta\in\Theta_p}\theta\sigma(\theta) \in \operatorname{conv}\Theta_p \subset [0,b]$, and so for all $\beta \geq b$, $\beta > \beta_p$. Then for all $p \in \mathcal{P}$, $E_{\langle\pi_p|\beta_p\rangle}\left[\left(\frac{\beta-\beta_p}{\beta_p(1-\beta_p)}\right)U(\beta)\right] \geq 0$. It follows from (EC-P) that D satisfies (IRθ-P) for each $\theta \in \Theta$.

Now whenever $\theta > \theta'$,

$$\begin{aligned} V(\theta,\theta|D) - V(\theta,\theta'|D) &= \sum_{\theta_i\in[\theta',\theta)}\left(V(\theta,\theta_{i+1}|D) - V(\theta,\theta_i|D)\right) \\ &= \sum_{\theta_i\in[\theta',\theta)}\begin{pmatrix} V(\theta_{i+1},\theta_{i+1}|D) - V(\theta_i,\theta_i|D) \\ +(\theta-\theta_{i+1})E_{\langle\pi_{p^*(\theta_{i+1})}|\beta_{p^*(\theta_{i+1})}\rangle}\left[\left(\frac{\beta-\beta_{p^*(\theta_{i+1})}}{\beta_{p^*(\theta_{i+1})}(1-\beta_{p^*(\theta_{i+1})})}\right)U(\beta)\right] \\ -(\theta-\theta_i)E_{\langle\pi_{p^*(\theta_i)}|\beta_{p^*(\theta_i)}\rangle}\left[\left(\frac{\beta-\beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})}\right)U(\beta)\right]\end{pmatrix} \text{ (by (7))} \\ &= \sum_{\theta_i\in[\theta',\theta)}(\theta-\theta_{i+1})\begin{pmatrix} E_{\langle\pi_{p^*(\theta_{i+1})}|\beta_{p^*(\theta_{i+1})}\rangle}\left[\left(\frac{\beta-\beta_{p^*(\theta_{i+1})}}{\beta_{p^*(\theta_{i+1})}(1-\beta_{p^*(\theta_{i+1})})}\right)U(\beta)\right] \\ -E_{\langle\pi_{p^*(\theta_i)}|\beta_{p^*(\theta_i)}\rangle}\left[\left(\frac{\beta-\beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})}\right)U(\beta)\right]\end{pmatrix} \text{ (by (EC-P))} \\ &\geq 0 \text{ (by (M-P))}; \end{aligned}$$

and whenever $\theta < \theta'$,

$$\begin{aligned}
V(\theta,\theta|D) - V(\theta,\theta'|D) &= \sum_{\theta_i \in [\theta,\theta')} (-V(\theta,\theta_{i+1}) + V(\theta,\theta_i)) \\
&= \sum_{\theta_i \in [\theta,\theta')} \begin{pmatrix} -V(\theta_{i+1},\theta_{i+1}|D) + V(\theta_i,\theta_i|D) \\ +(\theta_{i+1}-\theta) E_{\langle \pi_{p^*(\theta_{i+1})} | \beta_{p^*(\theta_{i+1})} \rangle} \left[\left(\frac{\beta - \beta_{p^*(\theta_{i+1})}}{\beta_{p^*(\theta_{i+1})}(1-\beta_{p^*(\theta_{i+1})})} \right) U(\beta) \right] \\ -(\theta_i - \theta) E_{\langle \pi_{p^*(\theta_i)} | \beta_{p^*(\theta_i)} \rangle} \left[\left(\frac{\beta - \beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})} \right) U(\beta) \right] \end{pmatrix} \text{ (by (7))} \\
&= \sum_{\theta_i \in [\theta,\theta')} (\theta_{i+1}-\theta) \begin{pmatrix} E_{\langle \pi_{p^*(\theta_{i+1})} | \beta_{p^*(\theta_{i+1})} \rangle} \left[\left(\frac{\beta - \beta_{p^*(\theta_{i+1})}}{\beta_{p^*(\theta_{i+1})}(1-\beta_{p^*(\theta_{i+1})})} \right) U(\beta) \right] \\ -E_{\langle \pi_{p^*(\theta_i)} | \beta_{p^*(\theta_i)} \rangle} \left[\left(\frac{\beta - \beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})} \right) U(\beta) \right] \end{pmatrix} \text{ (by (EC-P))} \\
&\geq 0 \text{ (by (M-P))}.
\end{aligned}$$

It follows that D satisfies (ICθ-P) for each $\theta \in \Theta$. Hence, D is implementable.
(ii): By (ICθ-P) and (7), for any θ, θ',

$$0 \leq V(\theta,\theta|D) - V(\theta,\theta'|D) = \begin{array}{l} V(\theta,\theta|D) - V(\theta',\theta'|D) \\ -(\theta-\theta') E_{\langle \pi_{p^*(\theta')} | \beta_{p^*(\theta')} \rangle} \left[\left(\frac{\beta - \beta_{p^*(\theta')}}{\beta_{p^*(\theta')}(1-\beta_{p^*(\theta')})} \right) U(\beta) \right] \end{array}$$

Swapping the labels θ and θ' and multiplying by -1 yields

$$V(\theta,\theta|D) - V(\theta',\theta'|D) \leq (\theta-\theta') E_{\langle \pi_{p^*(\theta)} | \beta_{p^*(\theta)} \rangle} \left[\left(\frac{\beta - \beta_{p^*(\theta)}}{\beta_{p^*(\theta)}(1-\beta_{p^*(\theta)})} \right) U(\beta) \right]$$

It follows that for any θ', θ,

$$(\theta-\theta') \left(E_{\langle \pi_{p^*(\theta)} | \beta_{p^*(\theta)} \rangle} \left[\left(\frac{\beta - \beta_{p^*(\theta)}}{\beta_{p^*(\theta)}(1-\beta_{p^*(\theta)})} \right) U(\beta) \right] - E_{\langle \pi_{p^*(\theta')} | \beta_{p^*(\theta')} \rangle} \left[\left(\frac{\beta - \beta_{p^*(\theta')}}{\beta_{p^*(\theta')}(1-\beta_{p^*(\theta')})} \right) U(\beta) \right] \right) \geq 0.$$

Hence, D must satisfy (M-P).
(iii): We begin by proving a claim stronger than (iii).

Lemma 7. *Let $D = \{\pi_p\}_{p \in \mathcal{P}} \cup \{\pi_0\}$ be implementable; let $\{a_k\}_0^K$ be as in Corollary 2, and $\mathcal{P} = \{0, \ldots, K\}$. For all $k \in \mathcal{P}$, there exists a menu $D^k = \{\pi_p^k\}_{p \in \mathcal{P}} \cup \{\pi_0\}$ such that $E_{\langle \pi_p^k | \beta_p \rangle}[W(\beta)] \geq E_{\langle \pi_p | \beta_p \rangle}[W(\beta)]$ for each $p \in \mathcal{P}$ that satisfies (M-P); satisfies (ECθ-P) for each $\theta \leq a_k$; and satisfies (ICθ-P) for $\theta > a_k$.*

Proof. We proceed by induction.

First observe that for any menu $D' = \{\pi'_p\}_{p\in\mathcal{P}} \cup \{\pi_0\}$ and any $\theta, \theta' \in \Theta_k$ for some $k \in \mathcal{P}$,

$$\begin{aligned} V(\theta,\theta|D') &= E_{\langle \pi'_k|\beta_k\rangle}\left[\left(\frac{\theta}{\beta_k}\beta + \frac{1-\theta}{1-\beta_k}(1-\beta)\right)U(\beta)\right] - C(\pi'_k) \\ &= V(\theta,\theta'|D') \qquad (8) \\ &= V(\theta',\theta'|D') + (\theta-\theta')E_{\langle \pi'_k|\beta_k\rangle}\left[\left(\frac{\beta-\beta_k}{\beta_k(1-\beta_k)}\right)U(\beta)\right]. \qquad (9) \end{aligned}$$

Initial step ($k=1$)*:* Since D is implementable, by (ii), it must satisfy (M-P). Then if D satisfies (ECθ-P) for all $\theta \in \Theta_1$ then we are done by setting $D^1 = D$. Suppose that D does not satisfy (ECθ-P) for some $\theta \in \Theta_1$. Then by (9), D does not satisfy (ECθ-P) for $\theta = \theta_0$. Since D is implementable, it must satisfy (IRθ-P) for $\theta = \theta_0$. Then we must have $E_{\langle \pi_1|\beta_1\rangle}\left[\left(\frac{\theta_0}{\beta_1}\beta + \frac{1-\theta_0}{1-\beta_1}(1-\beta)\right)U(\beta)\right] - C(\pi_1) > 0$.

Then by Lemma 10, there exists an experiment π' such that

$$E_{\langle \pi'|\beta_1\rangle}\left[\left(\frac{\theta_0}{\beta_1}\beta + \frac{1-\theta_0}{1-\beta_1}(1-\beta)\right)U(\beta)\right] - C(\pi') = 0 \qquad (10)$$

$$E_{\langle \pi'|\beta_1\rangle}\left[\left(\frac{\beta-\beta_1}{\beta_1(1-\beta_1)}\right)U(\beta)\right] = E_{\langle \pi_1|\beta_1\rangle}\left[\left(\frac{\beta-\beta_1}{\beta_1(1-\beta_1)}\right)U(\beta)\right]; \qquad (11)$$

$$E_{\langle \pi'|\beta_1\rangle}[W(\beta)] \geq E_{\langle \pi_1|\beta_1\rangle}[W(\beta)]. \qquad (12)$$

Now construct $D^1 = \{\pi^1_p\} \cup \{\pi_0\}$ by letting $\pi^1_1 = \pi'$, and $\pi^1_p = \pi_p$ for all $p \neq p^*(\theta_n)$. It follows from (12) that $E_{\langle \pi^1_p|\beta_p\rangle}[W(\beta)] \geq E_{\langle \pi_p|\beta_p\rangle}[W(\beta)]$ for each $p \in \mathcal{P}$. Moreover, by (11) and since D satisfies (M-P), D^1 satisfies (M-P). By (10) and (9), D^1 satisfies (EC-P) for all $\theta \leq a_1$.

Then by (9) and since (13) is strict, $V(\theta',\theta'|D^1) < V(\theta',\theta'|D)$ for all $\theta' \in \Theta_1$. Hence, from (7) and (11), $V(\theta,\theta'|D^1) < V(\theta,\theta'|D)$ for all $\theta' \in \Theta_1$ and $\theta > a_1$. Then since $V(\theta,\theta'|D^1) = V(\theta,\theta'|D)$ for all $\theta,\theta' \notin \Theta_1$, D^1 satisfies (ICθ-P) for all $\theta > a_1$.

Induction step: Suppose that $k > 1$, and that $D^{k-1} = \{\pi^{k-1}_p\}_{p\in\mathcal{P}} \cup \{\pi_0\}$ satisfies (M-P); satisfies (ECθ-P) for $\theta \leq a_k$; satisfies (ICθ-P) for $\theta > a_k$; and $E_{\langle \pi^{k-1}_p|\beta_p\rangle}[W(\beta)] \geq E_{\langle \pi_p|\beta_p\rangle}[W(\beta)]$ for each $p \in \mathcal{P}$. If D^{k-1} satisfies (ECθ-P) for each $\theta \in \Theta_k$ then we are done by setting $D^k = D^{k-1}$. Suppose that D^{k-1} does not satisfy (ECθ-P) for some $\theta \in \Theta_k$. Then by (9), D^{k-1} does not satisfy (ECθ-P) for $\theta = \theta_n$, where $\theta_n = \min \Theta_k$.

Since D^{k-1} satisfies (ICθ-P) for all $\theta \in \Theta_k$, we must have

$$
\begin{aligned}
& V(\theta_n, \theta_n | D^{k-1}) \geq V(\theta_n, \theta_{n-1} | D^{k-1}) \\
\Leftrightarrow\ & E_{\langle \pi_k^{k-1} | \beta_{p^*(\theta_n)} \rangle} \left[\left(\tfrac{\theta_n}{\beta_k} \beta + \tfrac{1-\theta_n}{1-\beta_k} (1-\beta) \right) U(\beta) \right] - C(\pi_k^{k-1}) \\
& \qquad \geq E_{\langle \pi_{k-1}^{k-1} | \beta_{k-1} \rangle} \left[\left(\tfrac{\theta_n}{\beta_{k-1}} \beta + \tfrac{1-\theta_n}{1-\beta_{k-1}} (1-\beta) \right) U(\beta) \right] - C(\pi_{k-1}^{k-1}) \qquad (13) \\
& \qquad = V(\theta_{n-1}, \theta_{n-1} | D^{k-1}) + (\theta_n - \theta_{n-1}) E_{\langle \pi_{k-1}^{k-1} | \beta_{k-1} \rangle} \left[\left(\tfrac{\beta - \beta_{k-1}}{\beta_{k-1}(1-\beta_{k-1})} \right) U(\beta) \right] \\
& \qquad = \sum_{i=0}^{n-1} (\theta_{i+1} - \theta_i) E_{\langle \pi_{p^*(\theta_i)}^{k-1} | \beta_{p^*(\theta_i)} \rangle} \left[\left(\tfrac{\beta - \beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})} \right) U(\beta) \right].
\end{aligned}
$$

Then since D^{k-1} does not satisfy (ECθ-P) for $\theta = \theta_n$, these inequalities must be strict. Then by Lemma 10, there exists an experiment π' such that

$$
\begin{aligned}
& E_{\langle \pi' | \beta_k \rangle} \left[\left(\tfrac{\theta_n}{\beta_k} \beta + \tfrac{1-\theta_n}{1-\beta_k} (1-\beta) \right) U(\beta) \right] - C(\pi') \\
& \qquad = \sum_{i=0}^{n-1} (\theta_{i+1} - \theta_i) E_{\langle \pi_{p^*(\theta_i)}^{n-1} | \beta_{p^*(\theta_i)} \rangle} \left[\left(\tfrac{\beta - \beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})} \right) U(\beta) \right]; \qquad (14)
\end{aligned}
$$

$$
E_{\langle \pi' | \beta_{p^*(\theta_n)} \rangle} \left[\left(\tfrac{\beta - \beta_{p^*(\theta_n)}}{\beta_{p^*(\theta_n)}(1-\beta_{p^*(\theta_n)})} \right) U(\beta) \right] = E_{\langle \pi_{p^*(\theta_n)}^{n-1} | \beta_{p^*(\theta_n)} \rangle} \left[\left(\tfrac{\beta - \beta_{p^*(\theta_n)}}{\beta_{p^*(\theta_n)}(1-\beta_{p^*(\theta_n)})} \right) U(\beta) \right]; \qquad (15)
$$

$$
E_{\langle \pi' | \beta_{p^*(\theta_n)} \rangle}[W(\beta)] \geq E_{\langle \pi_{p^*(\theta_n)}^{n-1} | \beta_{p^*(\theta_n)} \rangle}[W(\beta)] \geq E_{\langle \pi_{p^*(\theta_n)} | \beta_{p^*(\theta_n)} \rangle}[W(\beta)]. \qquad (16)
$$

Now construct $D^k = \{\pi_p^k\} \cup \{\pi_0\}$ by letting $\pi_{p^*(\theta_n)}^k = \pi'$, and $\pi_p^k = \pi_p^{k-1}$ for all $p \neq p^*(\theta_n)$. It follows from (16) and the induction hypothesis that $E_{\langle \pi_p^{k-1} | \beta_p \rangle}[W(\beta)] \geq E_{\langle \pi_p | \beta_p \rangle}[W(\beta)]$ for each $p \in \mathcal{P}$. Moreover, by (15) and the induction hypothesis, D^k satisfies (M-P). By (14) and (9), D^k satisfies (EC-P) for all $\theta \in \Theta_k$, so by the induction hypothesis, D^k satisfies (EC-P) for all $\theta \leq a_k$.

Then by (9) and since (13) is strict, $V(\theta', \theta' | D^k) < V(\theta', \theta' | D^{k-1})$ for all $\theta' \in \Theta_k$. Hence, from (7) and (15), $V(\theta, \theta' | D^k) < V(\theta, \theta' | D^{k-1})$ for all $\theta' \in \Theta_k$ and $\theta > a_k$. Then since $V(\theta, \theta' | D^k) = V(\theta, \theta' | D^{k-1})$ for all $\theta, \theta' \notin \Theta_k$, D^k satisfies (ICθ-P) for all $\theta > a_k$.

The lemma follows by induction. □

By Lemma 7, there exists a menu $\hat{D} = \{\hat{\pi}_p\}_{p \in \mathcal{P}} \cup \{\pi_0\}$ that satisfies (EC-P) and (M-P) such that $E_{\langle \hat{\pi}_p | \beta_p \rangle}[W(\beta)] \geq E_{\langle \pi_p | \beta_p \rangle}[W(\beta)]$ for each $p \in \mathcal{P}$. Then by (i), $\hat{D}$ is implementable. The claim (iii) follows. □

Proof of Proposition 1 (Implementable Menus) Follows immediately from Lemma 6. $\square$

Lemma 8. *Let $\theta \in \Theta$ and suppose that π is binary. Then*

i. $\langle \pi|\alpha\rangle([b,1]) = \begin{cases} \alpha\eta(\pi) + \pi(\bar{s}|0), & \frac{\pi(\bar{s}|1)\alpha}{\eta(\pi)\alpha+\pi(\bar{s}|0)} \geq b; \\ 0, & \text{otherwise.} \end{cases}$

ii. $E_{\langle \pi|\alpha\rangle}\left[\left(\frac{\theta}{\alpha}\beta + \frac{1-\theta}{1-\alpha}(1-\beta)\right)U(\beta)\right] = \begin{cases} (\theta\eta(\pi) + \pi_p(\bar{s}|0))u - C(\pi), & \frac{\pi(\bar{s}|1)\alpha}{\eta(\pi)\alpha+\pi(\bar{s}|0)} \geq b; \\ -C(\pi), & \text{otherwise}. \end{cases}$.

iii. If $E_{\langle \pi|\alpha\rangle}\left[\left(\frac{\theta}{\alpha}\beta + \frac{1-\theta}{1-\alpha}(1-\beta)\right)U(\beta)\right] \geq 0$ *and* $\pi \not\sim_B \pi_0$*, then* $\text{supp}\langle\pi|\alpha\rangle \cap [b,1] = \left\{\frac{\pi(\bar{s}|1)\alpha}{\alpha\eta(\pi)+\pi(\bar{s}|0)}\right\}$;

iv. $E_{\langle \pi_p|\beta_p\rangle}\left[\left(\frac{\beta-\beta_p}{\beta_p(1-\beta_p)}\right)U(\beta)\right] = \begin{cases} \eta(\pi_p), & \text{if } \frac{\pi(\bar{s}|1)\alpha}{\pi(\bar{s}|1)\alpha+\pi(\bar{s}|0)(1-\alpha)} \geq b; \\ 0, & \text{otherwise}. \end{cases}$.

Proof. (i) follows immediately from Lemma 1 and by Bayes' rule. (ii) then follows from (i). (iii) follows from (ii) since $C(\pi) > 0$ for all $\pi \not\sim_B \pi_0$ and by Bayes' rule. Finally, when $\frac{\pi(\bar{s}|1)\alpha}{\eta(\pi)\alpha+\pi(\bar{s}|0)} \geq b$, we have

$$E_{\langle \pi|\alpha\rangle}\left[\left(\frac{\beta-\alpha}{\alpha(1-\alpha)}\right)U(\beta)\right] = (\alpha\eta(\pi) + \pi(\bar{s}|0))\left(\frac{\frac{\pi(\bar{s}|1)\alpha}{\alpha\eta(\pi)+\pi(\bar{s}|0)} - \alpha}{\alpha(1-\alpha)}\right)u$$
$$= \left(\frac{\pi(\bar{s}|1) - \alpha\eta(\pi) - \pi(\bar{s}|0)}{1-\alpha}\right) = \eta(\pi),$$

as desired. $\square$

Proof of Proposition 2 (Youden's Index and Implementation) Let $D = \{\pi_\theta\}_{\theta\in\Theta} \cup \{\pi_0\}$ be a binary screening menu that is individually rational. Then each π_θ is binary and satisfies (IRθ). Claims (i) and (ii) thus follow immediately from Lemma 8 (iii) and (iv). $\square$

Proof of Theorem 1 (Screening Dominates Pooling) Let $D = \{\pi_p\}_{p\in\mathcal{P}} \cup \{\pi_0\}$ be an implementable pooling menu. By Lemmas 3 and 6(iii), it is without loss to let D be binary and satisfy (EC-P). By Corollary 2, there exist $\{a_k\}_0^K$ such that $a_k > a_{k-1}$ for each k, and for each $p \in \mathcal{P}$, $\Theta_p = (a_{k-1}, a_k] \cap \Theta$ for some k. Hence, it is without loss of generality to let $\mathcal{P} = \{1, \ldots, K\}$ and $\Theta_k = (a_{k-1}, a_k] \cap \Theta$ for each $k \in \mathcal{P}$; we write $E_\sigma[\theta|\theta \in \Theta_k] = \beta_k \in \Theta_k$.

Step 0: To begin, note that since D is implementable, for each $k \in \mathcal{P}$ and $\theta \in \Theta_k$, (IRθ-P) holds for $\pi_p = \pi_k$ and $\beta_p = \beta_k$, and so by Lemma 8 (iii), either $\pi_k \sim_B \pi_0$, or $\frac{\pi_k(\bar{s}|1)\beta_k}{\eta(\pi_k)\beta_k + \pi_k(\bar{s}|0)} \geq b$ and by Lemma 8 (ii) $(\eta(\pi_k)\theta + \pi_k(\bar{s}|0))u - C(\pi_k) \geq 0$.

Step I: First, we construct a binary screening menu $D' = \{\pi_\theta\}_{\theta \in \Theta} \cup \{\pi_0\}$ from D which is individually rational and leaves the principal weakly better off. For each $k \in \mathcal{P}$ and $\theta \in \Theta_k$, if $\frac{\pi_k(\bar{s}|1)\theta}{\eta(\pi_k)\theta + \pi_k(\bar{s}|0)} \geq b$ or $\pi_k \sim_B \pi_0$, set $\pi_\theta = \pi_k$. Otherwise, set $\pi_\theta = \pi_{k-1}$ (with the convention that $\pi_{k-1} \sim_B \pi_0$ when $k = 1$).

Fix $k \in \mathcal{P}$ and $\theta \in \Theta_k$. If $\pi_\theta = \pi_k$, then either $\pi_k \sim_B \pi_0$, or by Step 0 and Lemma 8(ii), $E_{\langle \pi_k|\theta\rangle}[U(\beta) - G(\beta|\theta)] = (\eta(\pi_k)\theta + \pi_k(\bar{s}|0))u - C(\pi_k) \geq 0$. If $\pi_\theta = \pi_{k-1}$, then by Step 0, either $\pi_{k-1} = \pi_0$ or (since $\theta > a_{k-1} \geq \beta_{k-1}$) $\frac{\pi_{k-1}(\bar{s}|1)\theta}{\pi_{k-1}(\bar{s}|1)\theta + \pi_{k-1}(\bar{s}|0)(1-\theta)} \geq \frac{\pi_{k-1}(\bar{s}|1)\beta_{k-1}}{\pi_{k-1}(\bar{s}|1)\beta_{k-1} + \pi_{k-1}(\bar{s}|0)(1-\beta_{k-1})} \geq b$. Then by Lemma 8(ii) and Step 0, for each $\theta_{k-1} \in \Theta_{k-1}$, $E_{\langle \pi_{k-1}|\theta\rangle}[U(\beta) - G(\beta|\theta)] = (\eta(\pi_{k-1})\theta + \pi_{k-1}(\bar{s}|0))u - C(\pi_{k-1}) \geq (\eta(\pi_{k-1})\theta_{k-1} + \pi_{k-1}(\bar{s}|0))u - C(\pi_{k-1}) \geq 0$. In all cases, π_θ satisfies (IRθ).

Now for each $k \in \mathcal{P}$ and $\theta \in \Theta_k$ such that $\pi_\theta \neq \pi_k$, either $\frac{\pi_k(\bar{s}|1)\theta}{\pi_k(\bar{s}|1)\theta + \pi_k(\bar{s}|0)(1-\theta)} < b$ or $\pi_k \sim_B \pi_0$. In either case, $E_{\langle \pi_k|\theta\rangle}[W(\beta)] = 0$. Hence, $\sum_{\theta \in \Theta} E_{\langle \pi_\theta|\theta\rangle}[W(\beta)] \geq \sum_{k=1}^{K} \sum_{\theta \in \Theta_k} E_{\langle \pi_k|\theta\rangle}[W(\beta)] \geq \sum_{k=1}^{K} E_{\langle \pi_\theta|\beta_k\rangle}[W(\beta)]$, where the last inequality follows from Blackwell's theorem.

Finally, since D is implementable, by Lemma 6(iii), it satisfies (M-P). Then by Lemma 8(iv), $\eta(\pi_k) \geq \eta(\pi_{k-1})$ for each $k \in \mathcal{P}$. Fix $k \in \mathcal{P}$ and $\theta \in \Theta_k$. If $\pi_\theta = \pi_k$, then by construction, $\eta(\pi_\theta) \geq \eta(\pi_{\theta'})$ for all $\theta \geq \theta'$. If $\pi_\theta = \pi_{k-1}$, then $\frac{\pi_k(\bar{s}|1)\theta}{\eta(\pi_k)\theta + \pi_k(\bar{s}|0)} < b$; then for all $\theta' \in \Theta_k$ with $\theta' < \theta$, $\frac{\pi_k(\bar{s}|1)\theta'}{\eta(\pi_k)\theta' + \pi_k(\bar{s}|0)} < b$, and so $\pi_{\theta'} = \pi_{k-1}$ as well. Then we once again have $\eta(\pi_\theta) \geq \eta(\pi_{\theta'})$ for all $\theta \geq \theta'$. Then by Proposition 2(i), D' satisfies (M-P).

Step II: Now, we construct a binary screening menu $\hat{D} = \{\hat{\pi}_\theta\}_{\theta \in \Theta\}} \cup \{\pi_0\}$ from D' which is individually rational and incentive compatible, and which the principal weakly prefers to D'. First, for each $\theta \in \Theta$ with $\pi_\theta \sim_B \pi_0$, set $\hat{\pi}_\theta = \pi_\theta$. For each such θ, since D' satisfies (M-P) and by Lemma 8(iv), we have $0 = \eta(\pi_{\theta_i})$, and thus $\pi_{\theta_i} \sim_B \pi_0$, for all $\theta_i \leq \theta$; it follows that $E_{\langle \hat{\pi}_\theta|\theta\rangle}[U(\beta)] - C(\hat{\pi}_\theta) = 0 = \sum_{\theta_i < \theta}(\theta_{i+1} - \theta_i)\eta(\hat{\pi}_{\theta_i})$.

Now define the mapping $k^* : \Theta \to \mathcal{P}$ such that for each $\theta \in \Theta$, $\theta \in \Theta_{k^*(\theta)}$, and fix $\theta \in \Theta$ such that $\pi_{k^*(\theta)} \neq \pi_0$. Since D satisfies (EC-P),

$$(\theta\eta(\pi_{k^*(\theta)}) + \pi_{k^*(\theta)}(\bar{s}|0))u - C(\pi_{k^*(\theta)}) = \sum_{\theta_i<\theta}(\theta_{i+1} - \theta_i)\eta(\pi_{k^*(\theta_i)})u,$$

by Step 0 and Lemma 8((ii)-(iii)). Since D is implementable, by Lemma 6(ii), it satisfies (M-P); then by Lemma 8(iv), $\eta(\pi_k) \geq \eta(\pi_{k-1})$ for each $k > 0$. Then by construction of D', $\eta(\pi_{\theta_i}) \leq \eta(\pi_{k^*(\theta_i)})$ for each $\theta_i \in \Theta$. Then we have

$$(\theta\eta(\pi_{k^*(\theta)}) + \pi_{k^*(\theta)}(\bar{s}|0))u - C(\pi_k) \geq \sum_{\theta_i<\theta}(\theta_{i+1} - \theta_i)\eta(\pi_{\theta_i})u. \tag{17}$$

Now fix $k \in \mathcal{P}$ and $\theta \in \Theta_k$ with $\pi_\theta \not\sim_B \pi_0$. If $\pi_\theta = \pi_{k-1}$, then by Step 1, for all $\theta_i < \theta$ with $\theta_i \in \Theta_k$, $\pi_{\theta_i} = \pi_{k-1}$ as well. Then applying (17) for $\theta' \in \Theta_{k-1}$ yields

$$(\theta\eta(\pi_{k-1}) + \pi_{k-1}(\bar{s}|0))u - C(\pi_k) \geq (\theta - \theta')\eta(\pi_{k-1}) + \sum_{\theta_i<\theta'}(\theta_{i+1} - \theta_i)\eta(\pi_{\theta_i})u$$

$$\Rightarrow E_{\langle\pi_\theta|\theta\rangle}[U(\beta)] - C(\pi_\theta) = (\theta\eta(\pi_{k-1}) + \pi_{k-1}(\bar{s}|0))u - C(\pi_{k-1}) \geq \sum_{\theta_i<\theta}(\theta_{i+1} - \theta_i)\eta(\pi_{\theta_i})u,$$

where the equality follows from Step 1. Alternatively, if $\pi_\theta = \pi_k$, then by Lemma 8(ii) and (17), $E_{\langle\pi_\theta|\theta\rangle}[U(\beta) - G(\beta|\theta)] = (\eta(\pi_k)\theta + \pi_k(\bar{s}|0))u - C(\pi_k) \geq \sum_{\theta_i<\theta}(\theta_{i+1} - \theta_i)\eta(\pi_{\theta_i})u$. In either case, by Lemma 10, we can choose a binary $\hat{\pi}_\theta$ such that $E_{\langle\hat{\pi}_\theta|\theta\rangle}[U(\beta)] - C(\hat{\pi}_\theta) = \sum_{\theta_i<\theta}(\theta_{i+1} - \theta_i)\eta(\pi_{\theta_i})u$; $\eta(\hat{\pi}_\theta) = \eta(\pi_\theta)$ (by Lemma 8(iv)); and $E_{\langle\hat{\pi}_\theta|\theta\rangle}[W(\beta)] \geq E_{\langle\pi_\theta|\theta\rangle}[W(\beta)]$.

Then since $\eta(\hat{\pi}_\theta) = \eta(\pi_\theta)$ for each $\theta \in \Theta$, it follows from Lemma 8 that $\hat{D}$ satisfies (ECθ), and (since D' satisfies (M(θ, θ'))) satisfies (M(θ, θ')). Then by Proposition 1, $\hat{D}$ is implementable. Moreover, we have $\sum_{\theta\in\Theta} E_{\langle\hat{\pi}_\theta|\theta\rangle}[W(\beta)] \geq \sum_{\theta\in\Theta} E_{\langle\pi_\theta|\theta\rangle}[W(\beta)] \geq \sum_{k=1}^{K} E_{\langle\pi_\theta|\beta_k\rangle}[W(\beta)]$, as desired. □

Lemma 9 (Uniqueness in (COPT′)). *If (COPT′) has a solution, it has a binary solution, and this solution is unique (up to Blackwell equivalence).*

Proof. Observe that $D = \{\pi^*_\theta\}_{\theta\in\Theta} \cup \{\pi_0\}$ solves (COPT′) if and only if $\pi^*_{\theta_n}$ solves (TBTθ_n) for each $\theta_n \in \Theta$. Since the objective in (TBTθ_n), $S(\beta, \theta_n) - R(\beta, \theta_n)$, satisfies the hypotheses of Lemma 11, if the problem $\max_\tau E_\tau[S(\beta, \theta_n) - R(\beta, \theta_n)]$ s.t. $E_\tau\beta = \theta_n$ has a solution π^*_n, it is unique, and $|\operatorname{supp} \pi^*_n| \leq 2$. Then the solution to (TBTθ_n) is unique up to Blackwell equivalence, and since $\pi^*_n = \langle\pi|\theta_n\rangle$ for some binary π, (TBTθ_n) has a binary solution. The claim follows. □

Proof of Theorem 2 (Existence and Uniqueness) This argument follows closely with the proof of Theorem 3 in Doval and Skreta (2023). Let V denote the value of (COPT). A priori, we do not know the value of (COPT) is attained, so

$$V = \sup_{\{\pi_\theta\}_{\theta\in\Theta}} \sum_{\theta\in\Theta} E_{\langle\pi_{\theta|\theta}\rangle}[W(\beta)]\sigma(\theta) \text{ s.t. } (EC), (M).$$

Denote by $f : [0,1]^\Theta \to \mathbb{R}$ the function $\{\beta_\theta\}_{\theta\in\Theta} \mapsto \sum_{n=0}^N W(\beta_{\theta_n})\sigma(\theta_n)$. Define a family of functions $\{\bar{g}_n\}_{n=0}^N$ by $\bar{g}_n : [0,1]^\Theta \to \mathbb{R}\cup\{-\infty\}$ where

$$\{\beta_\theta\}_{\theta\in\Theta} \mapsto U(\beta_{\theta_n}) - G(\beta_{\theta_n}|\theta_n) - \sum_{i<n}(\theta_{i+1}-\theta_i)\left(\frac{\beta_{\theta_i}-\theta_i}{\theta_i(1-\theta_i)}\right)U(\beta_{\theta_i})\ \forall\theta\in\Theta.$$

Define a family of functions $\{\bar{m}_n\}_{n=0}^N$ by $\bar{m} : [0,1]^\Theta \to \mathbb{R}$ where

$$\{\beta_\theta\}_{\theta\in\Theta} \mapsto \left(\frac{\beta_{\theta_n}-\theta_n}{\theta_n(1-\theta_n)}\right)U(\beta_{\theta_n}).$$

For a product distribution $\tau \in \Pi_{\theta\in\Theta}\Delta([0,1])$, the constraint (EC$\theta$) is satisfied if and only if $E_\tau\bar{g}_n = 0$ for all $n \in \{0,...,N\}$. Additionally, monotonicity is satisfied if and only if $E_\tau\bar{m}_n$ is non-decreasing in n. We'll decompose an element $y \in \mathbb{R}\times\mathbb{R}^\Theta\times\mathbb{R}^\Theta$ as $y = (y_p, (y_n)_{n=0}^N, (y_n^{\mathcal{M}})_{n=0}^N)$. Let

$$J = \{y \in \mathbb{R}\times\mathbb{R}^\Theta\times\mathbb{R}^\Theta : (\Theta, y) \in \text{conv}(Gr(f, \{\bar{g}_n\}_{n=0}^N, \{\bar{m}_n\}_{n=0}^N))\}.$$

That is, J is the set of possible point values of the objective and constraints.[30] The set of *pointwise feasible* vectors in J is $J_F = \{y \in J : y_n = 0\ \forall n \text{ and } y_n^{\mathcal{M}} \geq y_{n'}^{\mathcal{M}}\ \forall n \geq n'\}$. The condition $y_n = 0$ corresponds to the envelope condition holding for type θ_n and $y_n^{\mathcal{M}} \geq y_{n'}^{\mathcal{M}}\ \forall n' \leq n$ corresponds to the monotonicity condition for type θ_n.

To show the equivalence of (COPT) and (COPT′), we first define two distinct problems. We call the problem $\sup\{y_p : y \in J_F\}$ the **auxiliary primal**, and given two families of constants $\{\lambda_n\}_{n=0}^N$ and $\{\delta_n\}_{n=1}^N$, we call the problem

$$\sup_{y\in J}\left\{y_p + \sum_{n=0}^N \lambda_n y_n + \sum_{n=1}^N \delta_n(y_n^{\mathcal{M}} - y_{n-1}^{\mathcal{M}})\right\}$$

the **auxiliary dual**. Our argument establishes the following chain of equivalence

[30] See Rubin and Wesler (1958).

between problems:

$$\text{Primal (COPT)} \iff \text{Auxiliary Primal} \iff \text{Auxiliary Dual} \iff \text{Dual (COPT}')$$

Steps I-IV show that, for some $\{\lambda_n^*\}_{n=0}^N$ and $\{\delta_n^*\}_{n=1}^N$, the values of these problems coincide. Step V shows that a product distribution $\tau \in \Pi_{\theta\in\Theta}\Delta([0,1])$ solves (COPT) if and only if it solves (COPT′). Since the solution to (COPT′) exists, is unique, and is binary by Lemma 9, the solution to (COPT) also exists and is unique.

Step I (Show $\exists y, y' \in J_F$ with $y_p \neq y'_p$): We proceed constructively. First, observe that $y = (0, ..., 0) \in J_F$, since y is associated with the menu $D = \{\pi_\theta\}_{\theta\in\Theta} \cup \{\pi_0\}$ where each $\pi_\theta = \pi_0$. Next, consider the menu $D' = \{\pi'_\theta\}_{\theta\in\Theta} \cup \{\pi_0\}$ such that $\pi'_{\theta_n} = \pi_0$ for all $n < N$, and $\pi'_{\theta_N} \neq \pi_0$. Let π'_{θ_N} satisfy the envelope condition for type θ_N and let $\text{supp}\langle\pi'_{\theta_N}|\theta_N\rangle\} = \{\underline{\beta}, \overline{\beta}\}$ where $\underline{\beta} < b < \overline{\beta}$.[31] Since $y'_p = E_{\langle\pi'_{\theta_N}|\theta_N\rangle}[W(\beta)] > 0$, then $y'_p \neq y_p$.

Step II (Show $V = \sup\{y_p : y \in J_F\}$): We can show that the value of the problem (COPT) coincides with the value of $\sup\{y_p : y \in J_F\}$. First, let $\epsilon > 0$. Then $\exists y^\epsilon \in J$ such that (i) $y_p^\epsilon \geq \sup\{y_p : y \in J_F\} - \epsilon$, (ii) $y_n^\epsilon = 0$ for all n, and (iii) $y_n^{\epsilon,\mathcal{M}} \geq y_{n'}^{\epsilon,\mathcal{M}}$ for all $n \geq n'$. So there exists a set $\{(r_m, \{\beta_{\theta,m}\}_{\theta\in\Theta})\}_{m=1}^M$ such that $r = (r_m)_{m=1}^M \in \Delta(\bigcup_{m=1}^M \{\beta_{\theta,m}\}_{\theta\in\Theta})$, and $E_r\beta_{\theta_n} = \theta_n$ for all n, and

$$y_p^\epsilon = \sum_{m=1}^M r_m f(\{\beta_{\theta,m}\}_{\theta\in\Theta}) \geq \sup\{y_1 : y \in J_F\} - \epsilon; \quad y_n^\epsilon = \sum_{m=1}^M r_m \bar{g}_n(\{\beta_{\theta,m}\}_{\theta\in\Theta}) = 0 \ \forall n;$$

$$y_n^{\epsilon,\mathcal{M}} = \sum_{m=1}^M r_m \bar{m}_n(\{\beta_{\theta,m}\}_{\theta\in\Theta}) \text{ is non-decreasing in } n,$$

so r is feasible in (COPT). So $V \geq \sup\{y_p : y \in J_F\} - \epsilon$. Since this holds for all $\epsilon > 0$, $V \geq \sup\{y_p : y \in J_F\}$.

Next, we show that $\sup\{y_p : y \in J_F\} \geq V$. By Proposition 3(ii) in Yoder (2023) the solution to the auxiliary primal induces finitely many posteriors. That is, for $\epsilon > 0$, there exists a set $\{(r_m, \{\beta_{\theta,m}\}_{\theta\in\Theta})\}_{m=1}^M$ such that $r = (r_m)_{m=1}^M \in$

[31] Such an experiment exists, by assumption that $\exists\pi$ such that $E_{\langle\pi|\theta_N\rangle}[U(\beta, \theta_N)] - C(\pi) > 0$, then applying Lemma 10.

$\Delta(\bigcup_{m=1}^{M}\{\beta_{\theta,m}\}_{\theta\in\Theta})$, and $E_r\beta_{\theta_n} = \theta_n$ for all n, and

$$\sum_{m=1}^{M} r_m f(\{\beta_{\theta,m}\}_{\theta\in\Theta}) \geq V - \epsilon; \qquad \sum_{m=1}^{M} r_m \bar{g}_n(\{\beta_{\theta,m}\}_{\theta\in\Theta}) = 0 \;\forall n;$$

$$\sum_{m=1}^{M} r_m \bar{m}_n(\{\beta_{\theta,m}\}) \text{ is non-decreasing in } n.$$

Then, since $(\Theta, \sum_{m=1}^{M} r_m j(\{\beta_{\theta,m}\}_{\theta\in\Theta})) \in \mathrm{conv}(Gr\ j)$ where $j : [0,1]^\Theta \to \mathbb{R} \times \mathbb{R}^\Theta \times \mathbb{R}^\Theta$ is defined by $j = (f, \{\bar{g}_n\}_{n=0}^{N}, \{\bar{m}_n\}_{n=0}^{N})$ and $\sum_{m=1}^{M} r_m j(\{\beta_{\theta,m}\}_{\theta\in\Theta}) \in J_F$, then $\sup\{y_p : y \in J_F\} \geq V - \epsilon$; since this holds for all $\epsilon > 0$, $\sup\{y_p : y \in J_F\} \geq V$. It follows that $\sup\{y_p : y \in J_F\} = V$.

Step III (Show (COPT) *satisfies Slater's Condition):* We show that there exists a point $y \in J_F$ such that $y \in \mathrm{ri}(J)$, as in Assumption S of Doval and Skreta (2023). To show this, it is sufficient to show that there exists a point $y \in \mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J_F$ such that $y \in \mathrm{ri}(\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J)$, since $\mathrm{ri}(\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J) = \mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} \mathrm{ri}(J)$[32] and $\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J_F \cap \mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} \mathrm{ri}(J) \neq \emptyset$ implies $J_F \cap \mathrm{ri}(J) \neq \emptyset$.

Suppose for sake of contradiction that $\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J_F \cap \mathrm{ri}(\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J) = \emptyset$. By definition, J_F and J are convex; then so are $\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J_F$ and $\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J$, and hence $\mathrm{ri}(\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J)$. Then by the separating hyperplane theorem, there exists a nonzero $r = (r_p, \{r_n\}_{n=0}^{N}) \in \mathbb{R} \times \mathbb{R}^\Theta$ such that $r \cdot y \geq r \cdot x$ for all $y \in \mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J_F$ and $x \in \mathrm{ri}(\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J)$. Then $\mathrm{ri}(\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J) \subseteq \{x \in \mathbb{R} \times \mathbb{R}^\Theta : r \cdot x \leq \inf_{y \in \mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J_F} r \cdot y\}$; since the latter set is a closed half-space, $\{x \in \mathbb{R} \times \mathbb{R}^\Theta : r \cdot x \leq \inf_{y \in \mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J_F} r \cdot y\} \supseteq \mathrm{cl}(\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J) \supseteq \mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J$. Consequently,

$$\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} J_F \subseteq \arg\max_{y\in J} \langle r, \mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} y \rangle. \tag{18}$$

Observe that, by Step I, $\exists \hat{y}, \hat{y}' \in J_F$ such that $\hat{y}_p \neq \hat{y}'_p$. But since $\langle r, \mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} \hat{y} \rangle = \langle r, \mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} \hat{y}' \rangle$, $r_p \hat{y}_p = r_p \hat{y}'_p$. Since $\hat{y}_p \neq \hat{y}'_p$, $r_p = 0$.

Consider in particular a point $y \in J_F$ with $y_p < \sup\{y_p : y \in J_F\}$[33] and associate with this point a Bayes plausible distribution $\tau \in \Pi_{\theta\in\Theta}\Delta([0,1])$ with $y = (E_\tau f, \Pi_{n=0}^{N}\{0\}, \{E_\tau \bar{m}_\theta\}_{\theta\in\Theta})$. But by (18), we have $\mathrm{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} y \in \arg\max_{y\in J} \sum_{n=0}^{N} r_n y_n$.

Since r is nonzero and $r_p = 0$, we must have $r_n \neq 0$ for at least one $n \in \{0, \ldots, N\}$. We'll now construct a Bayes plausible product distribution $\tau' \in \Pi_{\theta\in\Theta}\Delta([0,1])$

[32]See, e.g., Proposition 2.1.12 of Hiriart-Urruty and Lemaréchal (2004).

[33]Which exists by Step I.

such that the associated $y' = (E_{\tau'} f, \{E_{\tau'}\bar{g}_n\}_{n=0}^N, \{E_{\tau'}\bar{m}_n\}_{n=0}^N)$ where $\text{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} y' \neq \text{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} y$ yields a higher value in the above program,inductively as follows: For each $n \in \{0, \ldots, N\}$, given $\text{proj}_{\theta<\theta_n}\tau'$, choose $\text{pro}_{\theta_n}\tau'$ such that the envelope condition (ECθ) for $\theta = \theta_n$ holds with the inequality "$>$" if $r_n > 0$ and "$<$" if $r_n < 0$, and choose $\text{proj}_{\theta_n}\tau' = \langle \pi_0|\theta_n\rangle$ if $r_n = 0$. Then $\sum_{n=0}^N r_n y'_n > \sum_{n=0}^N r_n y_n$, contradicting $\text{proj}_{\mathbb{R}\times\mathbb{R}^\Theta} y \in \arg\max_{y\in J} \sum_{n=0}^N r_n y_n$.

Step IV (Validate the Lagrangian Approach): The value $V = \sup\{y_p : y \in J_F\}$ coincides with the value of

$$\sup_{y\in J} y_p \ s.t. \begin{cases} y_n = 0, \ n \in \{0, ..., N\} \\ y_n^{\mathcal{M}} \geq y_{n'}^{\mathcal{M}} \ \forall n \geq n' \end{cases} .$$

Since (COPT) satisfies Slater's condition[34] and each constraint function is linear, by Rockafellar (1970) Corollary 28.2.2 and there exist weakly positive[35] Lagrange multipliers $\{\lambda_n^*\}_{n=0}^N$ and $\{\delta_n^*\}_{n=1}^N$ such that the value of the above program is equal to the maximized value (over J) of

$$L(y, \{\lambda_n^*\}_{n=0}^N, \{\delta_n^*\}_{n=1}^N) = y_p + \sum_{n=0}^N \lambda_n^* y_n + \sum_{n=1}^N \delta_n^*(y_n^{\mathcal{M}} - y_{n-1}^{\mathcal{M}}) \tag{19}$$

It follows that

$$\begin{aligned} V = \sup\{y_p : y \in J_F\} &= \sup_{y\in J}[y_p + \sum_{n=0}^N \lambda_n^* y_n + \sum_{n=1}^N \delta_n^*(y_n^{\mathcal{M}} - y_{n-1}^{\mathcal{M}})] \\ &= \sup_{\tau\in\Pi_{\theta\in\Theta}\Delta([0,1])} \left\{ E_\tau f + \sum_{n=0}^N \lambda_n^* E_\tau \bar{g}_n + \sum_{n=1}^N \delta_n^*(E_\tau \bar{m}_n - E_\tau \bar{m}_{n-1}) \ s.t. \ E_\tau \beta_{\theta_n} = \theta_n \ \forall n \right\} \\ &= \sup_{\tau\in\Pi_{\theta\in\Theta}\Delta([0,1])} \left\{ E_\tau \left[f + \sum_{n=0}^N \lambda_n^* \bar{g}_n + \sum_{n=1}^N \delta_n^*(\bar{m}_n - \bar{m}_{n-1}) \right] \ s.t. \ E_\tau \beta_{\theta_n} = \theta_n \ \forall n \right\} \end{aligned}$$

Step V (Coincidence of Solutions): Note first that τ is a solution to (COPT) if and only if $y_\tau = (E_\tau f, \{E_\tau \bar{g}_n\}_{n=0}^N, \{E_\tau \bar{m}_n\}_{n=0}^N\})$ solves the auxiliary primal. Additionally, τ solves (COPT') if and only if y_τ solves the auxiliary dual. All that remains is to show that y solves the auxiliary dual if and only if y solves the auxiliary primal. Since there is a unique solution to (COPT') by Lemma 9 and this solution is binary,

[34] There is a feasible point in the relative interior of a convex domain.
[35] By Lemma 14.

the auxiliary dual has a unique solution. By Rockafellar (1970) Corollary 28.1.1, the unique solution y to the auxiliary dual uniquely solves the auxiliary primal. Thus τ solves (COPT) if and only if τ solves (COPT′). The complementary slackness condition, (CS), follows from Rockafellar (1970) Theorem 28.3(a). Non-negativity of the Lagrange multipliers follows from Lemma 14.[36] □

Proof of Proposition 3 (Efficiency and the Social Planner's Problem) (If) Suppose that $\langle\pi|\theta\rangle$ solves (SPPθ). We consider three cases.

If $\lambda_a > 0$ and $\lambda_p > 0$, then there can be no $\pi' \neq \pi$ with $E_{\langle\pi'|\theta\rangle}[W(\beta)] \geq E_{\langle\pi|\theta\rangle}[W(\beta)]$ and $E_{\langle\pi'|\theta\rangle}[U(\beta) - G(\beta|\theta)] \geq E_{\langle\pi|\theta\rangle}[U(\beta) - G(\beta|\theta)]$ with one inequality strict; if there were, $\langle\pi'|\theta\rangle$ would achieve a higher value in (SPPθ). Thus, π is Pareto efficient for type θ.

If $\lambda_a = 0$, then (SPPθ) is equivalent to

$$\langle\pi|\theta\rangle \in \arg\max_{\tau\in\Delta(\Delta(\Omega))} \{E_\tau [W(\beta)] \text{ s.t. } E_\tau\beta = \theta\}. \tag{20}$$

Since W is convex, its concavification can be written $\overline{W}(\beta) = \beta w_1$. Hence, by Lemma 3 in Yoder (2022), the solution to (20) is the unique Bayes-plausible distribution with support $\{0, 1\}$, which is induced by the fully informative experiment π_∞. It follows that there can be no $\pi' \not\sim_B \pi_\infty$ with $E_{\langle\pi'|\theta\rangle}[W(\beta)] \geq E_{\langle\pi_\infty|\theta\rangle}[W(\beta)]$, and so π is Pareto efficient for type θ.

If $\lambda_p = 0$, then (SPPθ) is equivalent to

$$\langle\pi|\theta\rangle \in \arg\max_{\tau\in\Delta(\Delta(\Omega))} \{E_\tau [U(\beta) - G(\beta|\theta)] \text{ s.t. } E_\tau\beta = \theta\}. \tag{21}$$

By Lemma 11, the solution to (21) is unique. It follows that there can be no π' with $E_{\langle\pi'|\theta\rangle}[U(\beta) - G(\beta|\theta)] \geq E_{\langle\pi|\theta\rangle}[U(\beta) - G(\beta|\theta)]$, and so π is Pareto efficient for type θ.

(Only if) Suppose that π is Pareto efficient for type θ. If $E_{\langle\pi|\theta\rangle}[G(\beta|\theta)] = -\infty$ then (20) must hold, and so $\langle\pi|\theta\rangle$ solves (SPPθ) for $\lambda_a = 0$ and $\lambda_p > 0$. Suppose instead that $E_{\langle\pi|\theta\rangle}[G(\beta|\theta)] = y^* \in \mathbb{R}$, and let $E_{\langle\pi|\theta\rangle}[W(\beta)] = x^*$.

Let

$$Z_\theta = \{(x, y) \in \mathbb{R}^2 \mid \exists\tau \in \Delta(\Delta(\Omega)) : x \leq E_\tau [W(\beta)], E_\tau [U(\beta) - G(\beta|\theta)] \geq y, E_\tau\beta = \theta\}.$$

[36] Observe that by Lemma 10, replacing "=" with "≥" in (COPT) does not change the set of solutions.

Z_θ is convex: Let $(x, y), (x', y') \in Z_\theta$ and let $\lambda \in (0,1)$. Then for some τ, τ' with $E_\tau \beta = E_{\tau'} \beta = \theta$,

$$x \leq E_\tau \left[W(\beta)\right], \quad y \leq E_\tau \left[U(\beta) - G(\beta|\theta)\right], \quad x' \leq E_{\tau'} \left[W(\beta)\right], \quad y' \leq E_{\tau'} \left[U(\beta) - G(\beta|\theta)\right].$$

Let $\tau_\lambda = \lambda\tau + (1-\lambda)\tau'$. Then $E_{\tau_\lambda}\beta = \lambda E_\tau \beta + (1-\lambda) E_{\tau'} \beta = \theta$. Moreover,

$$\lambda x + (1-\lambda)x' \leq \lambda E_\tau \left[W(\beta)\right] + (1-\lambda) E_{\tau'} \left[W(\beta)\right] = E_{\tau_\lambda} \left[W(\beta)\right],$$
$$\lambda y + (1-\lambda)y' \leq \lambda E_\tau \left[U(\beta) - G(\beta|\theta)\right] + (1-\lambda) E_{\tau'} \left[U(\beta) - G(\beta|\theta)\right] = E_{\tau_\lambda} \left[U(\beta) - G(\beta|\theta)\right].$$

Hence $(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y') \in Z_\theta$.

Now $(x^*, y^*) \in \operatorname{Bd} Z_\theta$: If not, then there exists $\epsilon > 0$ such that the ϵ-ball about (x^*, y^*) is contained in Z_θ. But then $(x^* + \epsilon/2, y^* + \epsilon/2) \in Z_\theta$, a contradiction since π is Pareto efficient for type θ.

Then by the supporting hyperplane theorem, there exists $(\lambda_a^*, \lambda_p^*) \in \mathbb{R}^2 \setminus \{0\}$ such that $\lambda_a^* x^* + \lambda_p^* y^* \geq \lambda_a^* x + \lambda_p^* y$ for all $(x, y) \in Z_\theta$. Since $(x, y) \in Z_\theta$ implies $(x', y') \in Z_\theta$ for all $x' \leq x$, $y' \leq y$, it follows that $\lambda_a^*, \lambda_p^* \geq 0$.

If $\lambda_a^* > 0$, the claim follows immediately. If $\lambda_a^* = 0$, then

$$\begin{aligned} x^* &= \sup\{x \mid x = E_\tau \left[W(\beta)\right] \text{ for some } \tau \text{ with } E_\tau \left[U(\beta) - G(\beta|\theta)\right] > -\infty \text{ and } E_\tau \beta = \beta_0\} \\ &= \sup\{x \mid x = E_\tau \left[W(\beta)\right] \text{ for some } \tau \text{ with } \operatorname{supp} \tau \subseteq (0,1) \text{ and } E_\tau \beta = \beta_0\} \\ &= \beta_0 W(1) + (1-\beta_0) W(0) = \max_{\tau \in \Delta(\Delta(\Omega))} \{E_\tau[W(\beta)] \text{ s.t. } E_\tau \beta = \beta_0\}, \end{aligned}$$

where the third equality follows since W is convex and continuous, and since we have $\tau_n \to^{w^*} \tau_\infty$, where τ_∞ is the fully informative Bayes-plausible distribution over posteriors and

$$\tau_n\left(\{\beta_0/n\}\right) = 1 - \beta_0, \qquad \tau_n\left(\{1 - (1-\beta_0)/n\}\right) = \beta_0;$$
$$\tau_\infty\left(\{0\}\right) = 1 - \beta_0, \qquad \tau_\infty\left(\{1\}\right) = \beta_0.$$

The claim then follows by letting $\lambda_a = 0$ and $\lambda_p > 0$ in (SPPθ). □

Proof of Theorem 3 (Distortion Everywhere Else) Let $\theta = \theta_n$ for some $n < N$.

(ii): Suppose that π_{θ_n} is binary, is on the Pareto frontier for type θ_n, and is a Pareto improvement on $\pi^*_{\theta_n}$ for type θ_n. Since $\pi^*_{\theta_n}$ satisfies (IRθ) for type θ_n, and π_{θ_n} is a Pareto improvement on $\pi^*_{\theta_n}$, we must have $C(\pi_{\theta_n}) < \infty$.

Suppose toward a contradiction that $\pi_{\theta_n}(\underline{s}|1) \geq \pi^*_{\theta_n}(\underline{s}|1)$; then $\pi^*_{\theta_n} \not\succ_B \pi_{\theta_n}$. By Lemma 18, $\pi^*_{\theta_n} \not\succeq_B \pi_{\theta_n}$, so we must have $\pi_{\theta_n}(\underline{s}|0) > \pi^*_{\theta_n}(\underline{s}|0)$. Let $z = \max\{i \leq n \mid \delta_i = 0 \text{ or } i = 0\}$. That is, θ_z is the largest type less than or equal to θ_n such that the monotonicity constraint does not bind. Then by Lemma 5, $\pi^*_{\theta_n} = \pi^*_{\theta_z}$.

We show that there must be some $\pi' \in \Pi$ that is on the Pareto frontier for type θ_z for which $\pi'(\underline{s}|1) \geq \pi^*_{\theta_n}(\underline{s}|1)$, $\pi'(\underline{s}|0) > \pi^*_{\theta_n}(\underline{s}|0)$, and $C(\pi') < \infty$. If π_{θ_n} is on the Pareto frontier for type θ_z, this is immediate by setting $\pi' = \pi_{\theta_n}$; suppose not. Then there is some $\pi \in \Pi$ that is a Pareto improvement on π_{θ_n} for type θ_z; choose π' to be such a π that is on the Pareto frontier for type θ_z, and note that since $C(\pi_{\theta_n}) < \infty$, we must have $C(\pi') < \infty$ as well. If $\pi'(\underline{s}|1) < \pi_{\theta_n}(\underline{s}|1)$, then by Lemma 19, π' is also a Pareto improvement on π_{θ_n} for type θ_n, a contradiction since π_{θ_n} is on the Pareto frontier for type θ_n. So we must have $\pi'(\underline{s}|1) \geq \pi_{\theta_n}(\underline{s}|1) \geq \pi^*_{\theta_n}(\underline{s}|1)$. By Lemma 18, $\pi_{\theta_n} \not\succeq_B \pi'$, so $\pi'(\underline{s}|0) > \pi_{\theta_n}(\underline{s}|0) > \pi^*_{\theta_n}(\underline{s}|0)$.

Let $\text{supp}\langle \pi^*_{\theta_n}|\theta_z\rangle = \{\underline{\beta}^*, \overline{\beta}^*\}$ with $\underline{\beta}^* < \theta_z < \overline{\beta}^*$, and $\text{supp}\langle \pi'|\theta_z\rangle = \{\underline{\beta}', \overline{\beta}'\}$ with $\underline{\beta} < \theta_z < \overline{\beta}$. Since $\pi'(\underline{s}|1) \geq \pi^*_{\theta_n}(\underline{s}|1)$ and $\pi'(\underline{s}|0) > \pi^*_{\theta_n}(\underline{s}|0)$, π' and $\pi^*_{\theta_n}$ cannot be Blackwell-ranked. Then one of two cases must hold: (a) $\underline{\beta} \leq \underline{\beta}^*$ and $\overline{\beta} \leq \overline{\beta}^*$, with one inequality strict; or (b) $\underline{\beta} \geq \underline{\beta}^*$ and $\overline{\beta} \geq \overline{\beta}^*$, with one inequality strict.

Suppose that (b) holds. By Lemma 15, since $\theta_z \in \tilde{\Theta}$, we must have $\overline{\beta}^* > b$. By Proposition 3, since π' is on the Pareto frontier for type θ_z, there exist Pareto weights $\lambda_p, \lambda_a \geq 0$ such that $\langle \pi'|\theta_z\rangle$ solves (SPPθ) for $\theta = \theta_z$. Since $C(\pi') < \infty$, π' is not fully informative: $\pi' \not\sim_B \pi_\infty$. Then by Corollary 1, either equation (2) or (3) from Section 4.2 holds for $\theta = \theta_z$. Moreover, since $\pi^*_{\theta_n}$ solves (TBTθ_n) for $\theta = \theta_z$, (6) holds (with θ_n replaced by θ_z) at $(\underline{\beta}^*, \overline{\beta}^*)$. Then we have

$$\begin{aligned} R(b,\theta_z)/\lambda^*_z &= u - G(\overline{\beta}^*|\theta_z) - G'(\overline{\beta}^*|\theta_z) + G(\underline{\beta}^*|\theta_z) + G'(\underline{\beta}^*|\theta_z)(b - \underline{\beta}^*) \\ &< u - G(\overline{\beta}|\theta_z) - G'(\overline{\beta}|\theta_z) + G(\underline{\beta}|\theta_z) + G'(\underline{\beta}|\theta_z)(b - \underline{\beta}) = 0, \end{aligned} \tag{22}$$

but this contradicts Lemma 17. So it must be the case that (a) $\underline{\beta} \leq \underline{\beta}^*$ and $\overline{\beta} \leq \overline{\beta}^*$, with one inequality strict. Then since Bayesian updating is multiplicative in likelihood ratios, and $\overline{\beta} \leq \overline{\beta}^*$, we must have $\pi^*_{\theta_n}(\overline{s}|1)/\pi^*_{\theta_n}(\overline{s}|0) \geq \pi'(\overline{s}|1)/\pi'(\overline{s}|0)$. Since $\pi'(\underline{s}|0) > \pi^*_{\theta_n}(\underline{s}|0)$, we have $\pi'(\overline{s}|0) < \pi^*_{\theta_n}(\overline{s}|0)$, and hence $\pi^*_{\theta_n}(\overline{s}|1) > \pi'(\overline{s}|1)$, a contradiction.

(iii): Follows immediately from the assumption that there exists $\pi \in \Pi$ such that $E_{\langle \pi|\theta_n\rangle}[U(\beta)] - C(\pi) > 0$.

(i): We consider two cases.

Case 1: $\delta_n^* = 0$, or $n = 0$. Suppose toward a contradiction that $\pi_{\theta_n}^*$ is Pareto efficient for type θ_n. By Proposition 3, there exist $\lambda_p, \lambda_a \geq 0$, not identically zero, such that $\langle \pi_{\theta_n}^* | \theta_n \rangle$ solves (SPPθ).

If $\lambda_a = 0$, then $\langle \pi_{\theta_n}^* | \theta_n \rangle$ solves (20). Since W is convex, its concavification can be written $\overline{W}(\beta) = \beta w_1$. Hence, by Lemma 3 in Yoder (2022), the unique solution to (20) is the unique Bayes-plausible distribution with support $\{0, 1\}$. Then $\pi_{\theta_n}^*$ is Blackwell-equivalent to the fully informative binary experiment π_∞ with $\pi_\infty(\overline{s}|1) = \pi_\infty(\underline{s}|0) = 1$. But $E_{\langle \pi_\infty | \theta_n \rangle}[S(\beta, \theta_n) - R(\beta, \theta_n)] = -\infty < S(\theta_n, \theta_n) - R(\theta_n, \theta_n)$, so $\pi_{\theta_n}^*$ does not solve (TBTθ_n). Hence, D^* does not solve (COPT), contradicting Theorem 2.

If $\lambda_a > 0$, then by Lemma 11, we have $\operatorname{supp} \pi_{\theta_n}^* = \{\underline{\beta}^*, \overline{\beta}^*\}$ for $\underline{\beta}^*, \overline{\beta}^*$ that solve (2). And since π_θ^* solves (TBTθ_n), Lemma 11 implies that $\underline{\beta}^*, \overline{\beta}^*$ also solve (6). But by Lemma 17, $R(b, \theta_n)/\lambda_n^* > 0$; then $\underline{\beta}^*, \overline{\beta}^*$ cannot solve both (6) and (2), a contradiction.

Case 2: $n > 0$ and $\delta_n^* \neq 0$. Let $z = \max\{i \leq n \mid \delta_i = 0 \text{ or } i = 0\}$. That is, θ_z is the largest type below θ_n such that the monotonicity constraint does not bind. Then by Lemma 5, $\pi_{\theta_n}^* = \pi_{\theta_z}^*$.

By Case 1, $\pi_{\theta_z}^*$ is inefficient for type θ_z. Then there exists π_{θ_z} on the Pareto frontier for type θ_z which Pareto dominates $\pi_{\theta_z}^*$ for type θ_z, and by (ii), $\pi_{\theta_z}(\underline{s}|1) < \pi_{\theta_z}^*(\underline{s}|1)$. Then by Lemma 19, π_{θ_z} Pareto dominates $\pi_{\theta_z}^* = \pi_{\theta_n}^*$ for type θ_n as well. □

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Online Appendix

Technical Lemmas

Lemma 10 (Continuity Lemma). *If $\alpha < b$, $\langle \pi|\alpha\rangle \in \Delta([0,1])$, $m : [0,1] \to \mathbb{R}$ is a linear function, and $E_{\langle \pi|\alpha\rangle}[m(\beta)U(\beta) - G(\beta|\alpha)] > x$ for some $x \in \mathbb{R}_+$, then $\exists \pi' \in \Pi$ such that*

i. *π' is binary.*

ii. $E_{\langle \pi'|\alpha\rangle}[W(\beta)] > E_{\langle \pi|\alpha\rangle}[W(\beta)]$.

iii. $E_{\langle \pi'|\alpha\rangle}[m(\beta)U(\beta) - G(\beta|\alpha)] = x$.

iv. $E_{\langle \pi|\alpha\rangle}\left[\left(\frac{\beta-\alpha}{\alpha(1-\alpha)}\right)U(\beta)\right] = E_{\langle \pi'|\alpha\rangle}\left[\left(\frac{\beta-\alpha}{\alpha(1-\alpha)}\right)U(\beta)\right]$.

Proof. Let $\overline{\beta}^{\pi} = E_{\langle \pi|\alpha\rangle}[\beta|\beta \geq b]$ and $\underline{\beta}^{\pi} = E_{\langle \pi|\alpha\rangle}[\beta|\beta < b]$. Let $p = \langle \pi|\alpha\rangle([b,1])$ be the probability of approval for $\langle \pi|\alpha\rangle$; by Bayes' rule, $p\overline{\beta}^{\pi} + (1-p)\underline{\beta}^{\pi} = \alpha$. Then any binary experiment $\tilde{\pi}$ with $\text{supp}\langle \tilde{\pi}|\alpha\rangle = \{\underline{\beta}, \overline{\beta}\}$ such that $\underline{\beta} < \alpha < b \leq \overline{\beta}$ which satisfies

$$\left(\frac{\overline{\beta}-\alpha}{\alpha(1-\alpha)}\right)\left(\frac{\alpha-\underline{\beta}}{\overline{\beta}-\underline{\beta}}\right) = \left(\frac{\overline{\beta}^{\pi}-\alpha}{\alpha(1-\alpha)}\right)p \tag{23}$$

(and in particular, the binary experiment with $\text{supp}\langle \tilde{\pi}|\alpha\rangle = \{\underline{\beta}^{\pi}, \overline{\beta}^{\pi}\}$) will satisfy $E_{\langle \pi|\alpha\rangle}\left[\left(\frac{\beta-\alpha}{\alpha(1-\alpha)}\right)U(\beta)\right] = E_{\langle \tilde{\pi}|\alpha\rangle}\left[\left(\frac{\beta-\alpha}{\alpha(1-\alpha)}\right)U(\beta)\right]$. For such experiments, $\underline{\beta}$ can be defined implicitly as a function of $\overline{\beta}$, as follows:

$$\underline{\beta}(\overline{\beta}) = \frac{(\overline{\beta}-\alpha)\alpha - (\overline{\beta}^{\pi}p - \alpha p)\overline{\beta}}{\overline{\beta} - \alpha + \alpha p - \overline{\beta}^{\pi}p}.$$

This function is nondecreasing: we have

$$\underline{\beta}'(\overline{\beta}) = \frac{(\alpha+\alpha p-\overline{\beta}^{\pi}p)(\overline{\beta}-\alpha+\alpha p-\overline{\beta}^{\pi}p)-((\overline{\beta}-\alpha)\alpha-(\overline{\beta}^{\pi}p-\alpha p)\overline{\beta})}{(\overline{\beta}-\alpha+\alpha p-\overline{\beta}^{\pi}p)^2} = \frac{(\alpha p-\overline{\beta}^{\pi}p)^2}{(\overline{\beta}-\alpha+\alpha p-\overline{\beta}^{\pi}p)^2} \geq 0$$

Moreover,

$$\underline{\beta}(1) = \frac{(1-\alpha)\alpha - (\overline{\beta}^{\pi}p - \alpha p)}{1-\alpha+\alpha p - \overline{\beta}^{\pi}p} < \frac{(1-\alpha)\alpha - (\overline{\beta}^{\pi}p - \alpha p)\alpha}{1-\alpha+\alpha p - \overline{\beta}^{\pi}p} = \alpha.$$

Since we must have $\underline{\beta}(\overline{\beta}^{\pi}) = \underline{\beta}^{\pi}$, it follows that $\underline{\beta}([\overline{\beta}^{\pi}, 1]) \subseteq [\underline{\beta}^{\pi}, \alpha)$. Now define $f : [\overline{\beta}^{\pi}, 1] \to \mathbb{R} \cup \{-\infty\}$ by

$$f(\overline{\beta}) = \left(\frac{\alpha - \underline{\beta}(\overline{\beta})}{\overline{\beta} - \underline{\beta}(\overline{\beta})}\right)(m(\overline{\beta})u - G(\overline{\beta}|\alpha)) - \left(\frac{\overline{\beta} - \alpha}{\overline{\beta} - \underline{\beta}(\overline{\beta})}\right) G(\underline{\beta}(\overline{\beta})|\alpha).$$

Since $\lim_{\beta \to 0} G(\beta|\alpha) = \lim_{\beta \to 1} G(\beta|\alpha) = -\infty$, we have $\lim_{\overline{\beta} \to 1} f(\overline{\beta}) = -\infty$. Since $G(\cdot|\alpha)$ is strictly convex and m is linear, we have

$$\begin{aligned} f(\overline{\beta}^{\pi}) =& \langle \pi|\alpha \rangle([b,1]) \left(m\left(E_{\langle \pi|\alpha \rangle}[\beta|\beta \geq b]\right) u - G(E_{\langle \pi|\alpha \rangle}[\beta|\beta \geq b]|\alpha)\right) \\ &- \langle \pi|\alpha \rangle([0,b))G(E_{\langle \pi|\alpha \rangle}[\beta|\beta < b]|\alpha) \\ \geq& \langle \pi|\alpha \rangle([b,1]) \left(E_{\langle \pi|\alpha \rangle}[m(\beta)|\beta \geq b]u - E_{\langle \pi|\alpha \rangle}[G(\beta|\alpha)|\beta \geq b]\right) \\ &- \langle \pi|\alpha \rangle([0,b))E_{\langle \pi|\alpha \rangle}[G(\beta|\alpha)|\beta < b] \\ =& E_{\langle \pi|\alpha \rangle}\left[m(\beta)U(\beta) - G(\beta|\alpha)\right] > x. \end{aligned}$$

Since G and $\underline{\beta}(\cdot)$ are continuous, so is f. Then by the intermediate value theorem, there exists some $\overline{\beta}' > \overline{\beta}^{\pi}$ such that $f(\overline{\beta}') = x$. Then define $\pi' \in \Pi$ by $\mathrm{supp}\langle \pi'|\alpha \rangle = \{\underline{\beta}(\overline{\beta}'), \overline{\beta}'\}$, and observe that $E_{\langle \pi'|\alpha \rangle}[U(\beta) - G(\beta|\alpha)] = f(\overline{\beta}') = x$; hence, π' satisfies (iii) and (since it is binary) (i).

Since (23) holds for $(\underline{\beta}, \overline{\beta}) = (\underline{\beta}(\overline{\beta}'), \overline{\beta}')$, π' satisfies (iv). Moreover, we must have

$$\left(E_{\langle \pi'|\alpha \rangle}[\beta|\beta \geq b] - \alpha\right)\langle \pi'|\alpha \rangle([b,1]) = (\overline{\beta}' - \alpha)\frac{\alpha - \underline{\beta}(\overline{\beta}')}{\overline{\beta}' - \underline{\beta}(\overline{\beta}')} = \left(E_{\langle \pi|\alpha \rangle}[\beta|\beta \geq b] - \alpha\right)\langle \pi|\alpha \rangle([b,1]); \tag{24}$$

and furthermore, since $\overline{\beta}' > \overline{\beta}^{\pi}$, we must have

$$\begin{aligned} \langle \pi'|\alpha \rangle([b,1]) &= \frac{\alpha - \underline{\beta}(\overline{\beta}')}{\overline{\beta}' - \underline{\beta}(\overline{\beta}')} < p = \langle \pi|\alpha \rangle([b,1]); \\ &\Rightarrow \langle \pi'|\alpha \rangle([b,1])(\alpha - b) > \langle \pi|\alpha \rangle([b,1])(\alpha - b). \end{aligned} \tag{25}$$

Now observe that for any experiment $\hat{\pi}$, we can write

$$\begin{aligned} E_{\langle\hat{\pi}|\alpha\rangle}[W(\beta)] &= \langle\hat{\pi}|\alpha\rangle([b,1]) \cdot E_{\langle\hat{\pi}|\alpha\rangle}[W(\beta)|\beta \geq b] \\ &= \langle\hat{\pi}|\alpha\rangle([b,1]) \left(w_0 + (w_1 - w_0)E_{\langle\hat{\pi}|\alpha\rangle}[\beta|\beta \geq b]\right) \\ &= \langle\hat{\pi}|\alpha\rangle([b,1]) \left(\frac{-w_0}{b}\right) \left(E_{\langle\hat{\pi}|\alpha\rangle}[\beta|\beta \geq b] - b\right). \end{aligned}$$

Thus, adding (24) and (25) and multiplying both sides by $\frac{-w_0}{b} > 0$ yields

$$\begin{aligned} E_{\langle\pi'|\alpha\rangle}[W(\beta)] &= \langle\pi'|\alpha\rangle([b,1]) \left(\frac{-w_0}{b}\right) \left(E_{\langle\pi'|\alpha\rangle}[\beta|\beta \geq b] - b\right) \\ &> \langle\pi|\alpha\rangle([b,1]) \left(\frac{-w_0}{b}\right) \left(E_{\langle\pi|\alpha\rangle}[\beta|\beta \geq b] - b\right) = E_{\langle\pi|\alpha\rangle}[W(\beta)], \end{aligned}$$

and so π' satisfies (ii). □

Lemma 11 (A Necessary and Sufficient Tangent Line Condition). *Suppose that* $v : [0,1] \to \mathbb{R} \cup \{-\infty\}$ *is continuous and strictly concave on* $[0,b)$ *and on* $[b,1]$*, continuously differentiable on* $(0,b)$ *and* $(b,1)$*, that* $\lim_{\beta\to 0} v'(\beta) = \infty$*, and* $\lim_{\beta\to 1} v'(\beta) = -\infty$*, and that* $\alpha \in (0,b)$*. Then* $\max_\tau\{E_\tau v(\beta) \text{ s.t. } E_\tau\beta = \alpha\}$ *has a solution* τ^**, it is unique, and*

i. If there exist $\underline{\beta}, \overline{\beta}$ *with* $b, \alpha \in (\underline{\beta}, \overline{\beta})$ *and* $\frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}} = v'(\underline{\beta}) = v'(\overline{\beta})$*, then* $\operatorname{supp}\tau^* = \{\underline{\beta}, \overline{\beta}\}$*;*

ii. If not, and there exists $\underline{\beta} < \alpha < b$ *with* $\frac{v(b)-v(\underline{\beta})}{b-\underline{\beta}} = v'(\underline{\beta})$*, then* $\operatorname{supp}\tau^* = \{\underline{\beta}, b\}$*;*

iii. If the conditions in both (i) *and* (ii) *fail, then* $\operatorname{supp}\tau^* = \{\alpha\}$*.*

Proof. We begin by showing that Lemma 11 holds under the additional assumption that v is upper semicontinuous (or equivalently, $\lim_{\beta\uparrow b} v(\beta) \leq v(b)$). Since solutions to persuasion problems are preserved under upper semicontinuous hulls (Lemma S.5 in Yoder (2022)), the uniqueness of the solution to the upper semicontinuous hull problem guarantees that it is also the unique solution to the (possibly not upper semicontinuous) original problem. We then show existence by showing that if v is not upper semicontinuous, then the solution to the u.s.c. hull problem must be case (i) or (iii), in which case it is also a solution to the original problem.

To prove Lemma 11 under upper semicontinuity, we first show existence, and then show that the conditions in (i),(ii), and (iii) imply that τ^* has support $\{\underline{\beta}, \overline{\beta}\}$, $\{\underline{\beta}, b\}$, and $\{\alpha\}$, respectively. Then, we show that in each case, τ^* is the unique solution to $\max_\tau\{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$.

Existence of τ^* under upper semicontinuity. Follows from, e.g., Proposition 3 in Yoder (2023).

Support of τ^*. (i): Suppose that condition (i) holds, and v is upper semicontinuous. Then by strict concavity, $v(\underline{\beta}) + \frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}(\beta - \underline{\beta}) = v(\underline{\beta}) + v'(\underline{\beta})(\beta - \underline{\beta}) > v(\beta)$ for all $\beta \in [0, b)$, and $v(\underline{\beta}) + \frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}(\beta - \underline{\beta}) = v(\overline{\beta}) + v'(\overline{\beta})(\beta - \overline{\beta}) > v(\beta)$ for all $\beta \in [b, 1]$. Then by Proposition S.7 in Yoder (2022), the unique τ^* with $\operatorname{supp} \tau^* = \{\underline{\beta}, \overline{\beta}\}$ solves $\max_\tau\{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$.

(ii),(iii): Suppose that condition (i) fails, and v is upper semicontinuous. By Proposition 3 in Yoder (2023), $\max_\tau\{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$ has a solution τ^* with $|\operatorname{supp} \tau^*| \le 2$. Then by Proposition S.7 in Yoder (2022), either

A. $\operatorname{supp} \tau^* = \{\alpha\}$, or

B. $\operatorname{supp} \tau^* = \{\underline{\beta}', \overline{\beta}'\}$ and $v(\beta) \le v(\underline{\beta}') + \frac{v(\overline{\beta}')-v(\underline{\beta}')}{\overline{\beta}'-\underline{\beta}'}(\beta - \underline{\beta}')$ for all $\beta \in [0, 1]$.

Suppose that (B) holds. By Lemma 2 in Yoder (2022), we must have $\underline{\beta}' < \alpha < \overline{\beta}'$. Since v is strictly concave on $[0, b)$ and $[b, 1]$, we cannot have $\{\underline{\beta}', \overline{\beta}'\} \subset [0, b)$ or $\{\underline{\beta}', \overline{\beta}'\} \subset [b, 1]$. Then $\underline{\beta}' < \alpha, b \le \overline{\beta}'$. Moreover, by Lemma 12, since v is continuously differentiable on $(0, b)$ and $(b, 1)$, and $\lim_{\beta \to 0} v'(\beta) = \infty$ and $\lim_{\beta \to 0} v'(\beta) = -\infty$, we have $0 < \underline{\beta}' < \alpha, b \le \overline{\beta}' < 1$, and $\frac{v(\overline{\beta}')-v(\underline{\beta}')}{\overline{\beta}'-\underline{\beta}'} = v'(\beta)$ for each $\beta \in \{\underline{\beta}', \overline{\beta}'\}$ with $\beta \ne b$. Then since (i) fails, we must have $\overline{\beta}' = b$.

Then if condition (i) fails and (B) holds, condition (ii) follows. By contrapositive, if conditions (i) and (ii) both fail, we must have $\operatorname{supp} \tau^* = \{\alpha\}$; (iii) follows.

Now suppose that condition (i) fails and condition (ii) holds. Let τ' be such that $\operatorname{supp} \tau' = \{\underline{\beta}, b\}$ and $E_{\tau'} \beta = \alpha$. Since v is strictly concave on $[0, b)$, $E_{\tau'} v(\beta) = E_{\tau'} v(\beta) = v(\underline{\beta}) + \frac{v(b)-v(\underline{\beta})}{b-\underline{\beta}}(\alpha - \underline{\beta}) = v(\underline{\beta}) + v'(\underline{\beta})(\alpha - \underline{\beta}) > v(\alpha)$. Then $\operatorname{supp} \tau^* \ne \{\alpha\}$, and so (B) must hold. Then we must have $\operatorname{supp} \tau^* = \{\underline{\beta}', b\}$ for some $\underline{\beta}' < \alpha$. Suppose toward a contradiction that $\underline{\beta}' \ne \underline{\beta}$. Then since v is strictly concave on $[0, b)$, $v(\underline{\beta}') < v(\underline{\beta}) + v'(\underline{\beta})(\underline{\beta}' - \underline{\beta}) = v(\underline{\beta}) + \frac{v(b)-v(\underline{\beta})}{b-\underline{\beta}}(\underline{\beta}' - \underline{\beta})$. Then $E^*_\tau[v(\beta)] =$

$E^*_\tau[v(\beta)] < E^*_\tau\left[v(\underline{\beta}) + \frac{v(b)-v(\underline{\beta})}{b-\underline{\beta}}(\beta - \underline{\beta})\right] = v(\underline{\beta}) + \frac{v(b)-v(\underline{\beta})}{b-\underline{\beta}}(\alpha - \underline{\beta}) = E_{\tau'}[v(\beta)] = E_{\tau'}[v(\beta)]$, a contradiction since $\tau^* \in \arg\max_\tau\{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$. It follows that $\operatorname{supp}\tau^* = \{\underline{\beta}, b\}$.

Uniqueness of τ^*. First suppose that condition (i) holds, and $\operatorname{supp}\tau^* = \{\underline{\beta}, \overline{\beta}\}$ for $0 < \underline{\beta} < \alpha, b < \overline{\beta} < 1$. By strict concavity, $v(\underline{\beta}) + \frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}(\beta - \underline{\beta}) = v(\underline{\beta}) + v'(\underline{\beta})(\beta - \underline{\beta}) > v(\beta)$ for all $\beta \in [0, b)$, and $v(\underline{\beta}) + \frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}(\beta - \underline{\beta}) = v(\overline{\beta}) + v'(\overline{\beta})(\beta - \overline{\beta}) > v(\beta)$ for all $\beta \in [b, 1]$.

Then τ^* is the unique solution to $\max_\tau\{E_\tau v(\beta) \text{ s.t. } E_\tau\beta = \alpha\}$: Suppose that $\operatorname{supp}\tau \neq \{\underline{\beta}, \overline{\beta}\}$. If $|\operatorname{supp}\tau|$ is finite, then since $v(\beta) < v(\underline{\beta}) + \frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}(\beta - \underline{\beta})$ for all $\beta \notin \{\underline{\beta}, \overline{\beta}\}$, we have $E_\tau[v(\beta)] < E_{\tau^*}\left[v(\underline{\beta}) + \frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}(\beta - \underline{\beta})\right] = E_{\tau^*}[v(\beta)]$. Otherwise, either τ is nondegenerate on $[0, b)$ or it is nondegenerate on $[b, 1]$. Suppose the former; then since v is strictly concave on $[0, b)$ and $[b, 1]$, by Jensen's inequality, we have

$$
\begin{aligned}
E_\tau[v(\beta)|\beta < b] &< v(E_\tau[\beta|\beta < b]) \\
E_\tau[v(\beta)|\beta \geq b] &\leq v(E_\tau[\beta|\beta \geq b]) \\
\Rightarrow E_\tau[v(\beta)] = \begin{array}{l} E_\tau[v(\beta)|\beta < b]\tau([0,b)) \\ +E_\tau[v(\beta)|\beta \geq b]\tau([b,1]) \end{array} &< \begin{array}{l} v(E_\tau[\beta|\beta < b])\tau([0,b)) \\ +v(E_\tau[\beta|\beta \geq b])\tau([b,1]) \end{array} \\
&\leq \begin{array}{l} \left(v(\underline{\beta}) + \frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}(E_\tau[\beta|\beta < b] - \underline{\beta})\right)\tau([0,b)) \\ + \left(v(\underline{\beta}) + \frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}(E_\tau[\beta|\beta \geq b] - \underline{\beta})\right)\tau([b,1]) \end{array} \\
&= v(\underline{\beta}) + \frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}(\alpha - \underline{\beta}) = E_{\tau^*}[v(\beta)].
\end{aligned}
$$

A symmetric argument shows that $E_\tau[v(\beta)] < E_{\tau^*}[v(\beta)]$ when τ is non-degenerate on $[b, 1]$.

For case (ii), the same argument with $\overline{\beta} = b$ shows that τ^* is unique.

For case (iii), where $\operatorname{supp}\tau^* = \{\alpha\}$: First note that by Proposition 3 in Yoder (2023), $V(\alpha) = v(\alpha)$, and by Lemma 12, if $\alpha \neq b$, $v'(\alpha) = V'(\alpha)$. Suppose that $\tau' \in \arg\max_\tau\{E_\tau v(\beta) \text{ s.t. } E_\tau\beta = \alpha\}$ and $\operatorname{supp}\tau' \neq \{\alpha\}$. Then since $E_{\tau'}\beta = \alpha$, there must be some $\beta, \beta' \in \operatorname{supp}\tau'$ with $\beta < \alpha < \beta'$. If $\alpha < b$, then by Lemma

12, since v is differentiable and strictly concave on $(0,b)$, $v'(\beta) = V'(\alpha) = v'(\alpha) < v'(\beta)$, a contradiction. Alternatively, if $\alpha \geq b$, then by Lemma 12, since v is differentiable on $(b,1)$, $v'(\beta') = V'(\alpha)$. By Proposition 3 in Yoder (2023), V is affine on $\text{conv}(\text{supp}\,\tau') \supseteq [\alpha, \beta')$, so $v(\alpha) = V(\alpha) = V(\beta') + V'(\beta')(\alpha - \beta') = v(\beta') + v'(\beta')(\alpha - \beta')$, a contradiction since v is strictly concave on $(b,1)$.

Extension to non-upper semicontinuous v. Suppose that v is not upper semicontinuous. Then $\lim_{\beta\uparrow b} v(\beta) > v(b)$. Let $\bar{v}$ be the upper semicontinuous hull of v; we have $\bar{v}(b) = \lim_{\beta\uparrow b} v(\beta)$ and $\bar{v}(\beta) = v(\beta)$ for all $\beta \neq b$, so that $\bar{v}$ is strictly concave and continuous on $[0,b]$ and $(b,1]$. Since Lemma 11 holds under upper semicontinuity, there exists a unique $\tau^* \in \arg\max_\tau\{E_\tau \bar{v}(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$, and

i. If there exist $\underline{\beta}, \overline{\beta}$ with $b, \alpha \in (\underline{\beta}, \overline{\beta})$ and $\frac{v(\overline{\beta}) - v(\underline{\beta})}{\overline{\beta} - \underline{\beta}} = v'(\underline{\beta}) = v'(\overline{\beta})$, then $\text{supp}\,\tau^* = \{\underline{\beta}, \overline{\beta}\}$;

ii.' If not, and there exists $\underline{\beta} < \alpha$ with $\frac{\bar{v}(b) - v(\underline{\beta})}{b - \underline{\beta}} = v'(\overline{\beta})$, then $\text{supp}\,\tau^* = \{\underline{\beta}, b\}$;

iii.' If the conditions in both (i) and (ii') fail, then $\text{supp}\,\tau^* = \{\alpha\}$.

But (ii') cannot hold: Suppose it does. Then $\frac{\bar{v}(b) - \bar{v}(\underline{\beta})}{b - \underline{\beta}} = \frac{\bar{v}(b) - v(\underline{\beta})}{b - \underline{\beta}} = \bar{v}'(\overline{\beta})$, a contradiction since $\bar{v}$ is strictly concave and continuous on $[0,b]$.

Thus, either (i) holds, or $\text{supp}\,\tau^* = \{\alpha\}$; in either case, $b \notin \text{supp}\,\tau^*$. Then by definition of $\bar{v}$, for all τ with $E_\tau \beta = \alpha$, we have

$$E_{\tau^*}[v(\beta)] = E_{\tau^*}[\bar{v}(\beta)] \geq E_\tau[\bar{v}(\beta)] \geq E_\tau[v(\beta)],$$

and so $\tau^* \in \arg\max_\tau\{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$. Hence $\arg\max_\tau\{E_\tau \bar{v}(\beta) \text{ s.t. } E_\tau \beta = \alpha\} \subseteq \arg\max_\tau\{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$; by By Lemma S.5 in Yoder (2022), the two sets are equal, and τ^* uniquely solves $\max_\tau\{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$.

It remains to be shown that (ii) cannot hold, and so τ^* is pinned down by (i),(ii), and (iii), rather than by (i),(ii'), and (iii'). Suppose that it does, and there exists $\underline{\beta} < \alpha < b$ with $v'(\underline{\beta}) = \frac{v(b) - v(\underline{\beta})}{b - \underline{\beta}}$. Since v is strictly concave on $[0,b)$, $v(\underline{\beta}) + v'(\underline{\beta})(b - \underline{\beta}) > \lim_{\beta\uparrow b} v(\beta) > v(b)$. Then $v'(\underline{\beta}) > \frac{v(b) - v(\underline{\beta})}{b - \underline{\beta}}$, a contradiction. $\square$

Lemma 12. *Suppose that* $v : [0,1] \to \mathbb{R} \cup \{-\infty\}$ *is upper semicontinuous and* $\alpha \in \text{ri}(\text{dom}\,v)$, *and that* $\tau^* \in \arg\max_\tau\{E_\tau v(\beta)$ *s.t.* $E_\tau \beta = \alpha\}$. *Then for any* $\beta \in \text{supp}\,\tau^*$,

i. If v *is differentiable at* β, $v'(\beta) = V'(\alpha)$.

ii. *If v is strictly concave on $[0,z)$ and $(1-z,1]$ and continuously differentiable on $(0,z)$ and $(1-z,1)$ for some $z>0$, $\lim_{x\to 0} v'(x)=\infty$, and $\lim_{x\to 0} v'(x)=-\infty$, then $\beta\notin\{0,1\}$.*

Proof. By Proposition 3 in Yoder (2023), $v(\beta)=V(\beta)$, and V is affine on $\operatorname{conv}(\operatorname{supp}\pi^*)$. Then $v(\beta)=V(\alpha)+V'(\alpha)(\beta-\alpha)$. Since V is concave, for all $\beta'\in[0,1]$ we have

$$\begin{aligned} v(\beta') &\le V(\alpha)+V'(\alpha)(\beta'-\alpha), \\ &= v(\beta)+V'(\alpha)(\beta'-\beta). \end{aligned} \tag{26}$$

Then for $0<\epsilon<\min\{\beta,1-\beta\}$,

$$\begin{aligned} v(\beta+\epsilon) &\le v(\beta)+V'(\alpha)\epsilon, \\ v(\beta-\epsilon) &\le v(\beta)-V'(\alpha)\epsilon, \\ \Rightarrow \frac{v(\beta+\epsilon)-v(\beta)}{\epsilon} &\le V'(\alpha)\le \frac{v(\beta)-v(\beta-\epsilon)}{\epsilon}. \end{aligned}$$

Then if v is differentiable at β, (i) follows by the squeeze theorem.

For (ii), suppose $\beta=0$, v is continuously differentiable on $(0,z)$ and strictly concave on $[0,z)$ for some $z>0$, and $\lim_{x\to 0} v'(x)=\infty$. Since $\lim_{x\to 0} v'(x)=\infty$, there exists $y\in(0,\min\{z,\alpha\})$ such that $v'(y)>V'(\alpha)$. Since v is strictly concave on $[0,z)$, $v(0)<v(y)-v'(y)y$. By Proposition 3 in Yoder (2023), $v(0)=V(0)$. And by definition, $V(y)\ge v(y)$. Then by (26), we have

$$V(0)+V'(\alpha)y=v(0)+V'(\alpha)y<v(0)+v'(y)y<v(0)+(v(y)-v(0))=v(y)\le V(y),$$

a contradiction since V is concave. A similar argument shows that $\beta\neq 1$. □

Lemma 13. *Suppose that $v:[0,1]\to\mathbb{R}\cup\{-\infty\}$ is concave, and is strictly concave about the threshold: For any $\beta<b<\beta'$ and any $x\in\partial v(\beta)$, $y\in\partial v(\beta')$, we have $x>y$. If $\operatorname{supp}\tau=\{\underline{\beta},\overline{\beta}\}$, $\operatorname{supp}\tau'=\{\underline{\beta}',\overline{\beta}'\}$, $E_\tau\beta=E_{\tau'}\beta$, $\underline{\beta}<b<\overline{\beta}$, and τ' is a mean-preserving spread of τ, then $E_{\tau'}[v(\beta)]<E_\tau[v(\beta)]$.*

Proof. Let $v_1(\beta)=v(\underline{\beta})+\frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}(\beta-\underline{\beta})$ be the secant line to v at $\underline{\beta}$ and $\overline{\beta}$. Then we have $E_{\tau'}[v_1(\beta)]=E_\tau[v_1(\beta)]=E_\tau[v(\beta)]$.

Since v is concave, for any $x\in\partial v(\underline{\beta})$ and $y\in\partial v(\overline{\beta})$, we have $x\ge\frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}}\ge y$. If either holds with equality, then since v is concave, it must be affine on $[\underline{\beta},\overline{\beta}]$ — a contradiction since $\underline{\beta} < b < \overline{\beta}$ and v is strictly concave about the threshold. So we must have $x > \frac{v(\overline{\beta})-v(\underline{\beta})}{\overline{\beta}-\underline{\beta}} > y$ for all $x \in \partial v(\underline{\beta})$ and $y \in \partial v(\overline{\beta})$. Choose such an x and y; since v is concave,

$$\begin{aligned} v(\overline{\beta}') &\leq v(\overline{\beta}) + y(\overline{\beta}' - \overline{\beta}) < v_1(\overline{\beta}'), && \text{if } \overline{\beta} \neq \overline{\beta}'; \\ v(\underline{\beta}') &\leq v(\underline{\beta}) + x(\underline{\beta}' - \underline{\beta}) < v_1(\underline{\beta}'), && \text{if } \underline{\beta} \neq \underline{\beta}'. \end{aligned}$$

Since τ and τ' are distinct, it follows that $E_{\tau'}[v(\beta)] < E_{\tau'}[v_1(\beta)] = E_\tau[v(\beta)]$, as desired. □

Lemma 14 (Non-negativity of Lagrange Multipliers). *Let $X \subseteq \mathbb{R}^n$, let $f : X \to \mathbb{R}$, and let $g : X \to \mathbb{R}$. Consider the following pair of optimization problems*

$$\max_{x \in X} f(x) \text{ s.t. } g_i(x) \geq 0 \ \forall i = 1, ..., m \tag{ICP}$$

$$\max_{x \in X} f(x) \text{ s.t. } \begin{cases} g_i(x) \geq 0 \ \forall i = 1, ..., r \\ g_i(x) = 0 \ i = r+1, ..., m \end{cases} \tag{ECP}$$

Suppose (ICP) *and* (ECP) *are ordinary convex programs (Rockafellar, 1970) satisfying the hypotheses of Rockafellar (1970) Theorem 28.2. If all $x^* \in X$ that solve* (ICP) *satisfy $g_i(x^*) = 0$ for all $i = r+1, ..., m$, then $\exists \lambda \in \mathbb{R}^m$ with $\lambda \geq 0$ such that the value of*

$$\max_{x \in X} f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \tag{ECL}$$

is equal to the value of (ECP).

Proof. Observe first that there exists a Kuhn-Tucker vector $\gamma \in \mathbb{R}^m$ with $\gamma \geq 0$ such that

$$\max_{x \in X} \{f(x) \text{ s.t. } g_i(x) \geq 0 \ \forall i = 1, ..., m\} = \max_{x \in X} \left\{ f(x) + \sum_{i=1}^{m} \gamma_i g_i(x) \right\}$$

by Theorem 28.2 of Rockafellar (1970). Next, since

$$\max_{x \in X} \{f(x) \text{ s.t. } g_i(x) \geq 0 \ \forall i = 1, ..., m\} = \max_{x \in X} \left\{ f(x) \text{ s.t. } \begin{cases} g_i(x) \geq 0 \ \forall i = 1, ..., r \\ g_i(x) = 0 \ i = r+1, ..., m \end{cases} \right\},$$

it follows immediately that

$$\max_{x\in X}\left\{f(x)\ s.t.\ \begin{cases} g_i(x)\geq 0\ \forall i=1,...,r \\ g_i(x)=0\ i=r+1,...,m\end{cases}\right\}=\max_{x\in X}\left\{f(x)+\sum_{i=1}^{m}\gamma_i g_i(x)\right\}.$$

Setting $\lambda=\gamma$ concludes the proof. □

Implementability

Proof of Lemma 1 (Agent's Distribution of Principal's Posteriors) Let $B\subseteq[0,1]$. If $\langle\pi|\alpha\rangle(B)=0$, the statement follows trivially. Otherwise, define $S_B\subseteq S$ to be the set of signal realizations from π that induce posterior beliefs $\beta\in B$ to a principal with prior α. Then the probability with which an agent of type θ believes that S_B will occur is

$$\theta\pi(S_B|\omega=1)+(1-\theta)\pi(S_B|\omega=0)=\frac{\theta}{\alpha}\cdot\alpha\pi(S_B|\omega=1)+\frac{1-\theta}{1-\alpha}\cdot(1-\alpha)\pi(S_B|\omega=0).$$

By Bayes' rule, note that

$$\int_B \beta d\langle\pi|\alpha\rangle(\beta)=\pi(S_B|\omega=1)\alpha;\quad 1-\int_B\beta d\langle\pi|\alpha\rangle(\beta)=\pi(S_B|\omega=0)(1-\alpha).$$

So, the probability with which an agent of type θ believes that S_B will occur simplifies to

$$\int_B\left(\frac{\theta}{\alpha}\beta+\frac{1-\theta}{1-\alpha}(1-\beta)\right)d\langle\pi|\alpha\rangle(\beta)$$

as desired. □

Proof of Lemma 2 (Pooling is Local) Let $D=\{\pi_p\}_{p\in\mathcal{P}}\cup\{\pi_0\}$ be an implementable menu. For any $p,p'\in\mathcal{P}$ such that $\text{conv}(\Theta_p)\cap\text{conv}(\Theta_{p'})\neq\emptyset$ (if no such pair p,p' exists, pooling is local in D), without loss of generality[37] there exists some $\theta_1,\theta_2\in\Theta_p$ and $\alpha\in(0,1)$ such that $\tilde{\theta}=\alpha\theta_1+(1-\alpha)\theta_2\notin\Theta_p$ and $\tilde{\theta}\in\Theta_{p'}$. Define the function

$$Q_p(\theta)=E_{\langle\pi_p|\beta_p\rangle}\left[\left(\frac{\theta}{\beta_p}\beta+\frac{1-\theta}{1-\beta_p}(1-\beta)\right)U(\beta)-G(\beta|\beta_p)\right].$$

[37] If this is not the case, there exists some $\theta_1,\theta_2\in\Theta_{p'}$ and $\alpha\in(0,1)$ such that $\tilde{\theta}=\alpha\theta_1+(1-\alpha)\theta_2\notin\Theta_{p'}$ and $\tilde{\theta}\in\Theta_p$. The argument is identical.

Clearly, Q_p is an affine function of θ. Then, by incentive compatibility (ICθ-P) for $\theta = \tilde{\theta}$,

$$Q_{p'}(\alpha\theta_1 + (1-\alpha)\theta_2) \geq Q_p(\alpha\theta_1 + (1-\alpha)\theta_2).$$

Incentive compatibility also implies that $Q_p(\theta_1) \geq Q_{p'}(\theta_1)$ and $Q_p(\theta_2) \geq Q_{p'}(\theta_2)$. Then since Q is affine,

$$\begin{aligned} \alpha Q_{p'}(\theta_1) + (1-\alpha)Q_{p'}(\theta_2) &= Q_{p'}(\alpha\theta_1 + (1-\alpha)\theta_2) \\ &\geq Q_p(\alpha\theta_1 + (1-\alpha)\theta_2) \\ &= \alpha Q_p(\theta_1) + (1-\alpha)Q_p(\theta_2) \geq \alpha Q_{p'}(\theta_1) + (1-\alpha)Q_{p'}(\theta_2) \end{aligned}$$

so these inequalities all hold with equality. Since Q_p and $Q_{p'}$ are affine, $Q_p(\theta) = Q_{p'}(\theta)$ for all $\theta \in [0,1]$ (since Q_p and $Q_{p'}$ are affine, we can extend their domains to $[0,1]$). Next, denote $p_1 = Pr_{\langle \pi_p | \beta_p \rangle}(\beta \geq b | \omega = 1)$, $p_0 = Pr_{\langle \pi_p | \beta_p \rangle}(\beta \geq b | \omega = 0)$, $p_1' = Pr_{\langle \pi_{p'} | \beta_{p'} \rangle}(\beta \geq b | \omega = 1)$, and $p_0' = Pr_{\langle \pi_{p'} | \beta_{p'} \rangle}(\beta \geq b | \omega = 0)$. Since $Q_p(\theta) = Q_{p'}(\theta)$ for all $\theta \in [0,1]$,

$$(\theta p_1 + (1-\theta)p_0)u - C(\pi_p) = (\theta p_1' + (1-\theta)p_0') - C(\pi_{p'}).$$

Plugging in $\theta = 1$ and $\theta = 0$ yields the following expressions:

$$p_1 - p_1' = \frac{C(\pi_p) - C(\pi_{p'})}{u}; \qquad p_0 - p_0' = \frac{C(\pi_p) - C(\pi_{p'})}{u}.$$

Observe that, when π_p is conducted by a type-θ agent, and the principal updates from β_p, the principal's expected payoff is $R_p(\theta) = \theta p_1 w_1 + (1-\theta)p_0 w_0$, which is affine in θ. Moreover,

$$R_p(\theta) - R_{p'}(\theta) = \theta w_1(p_1 - p_1') + (1-\theta)w_0(p_0 - p_0') = \left(\frac{C(\pi_{p'}) - C(\pi_p)}{u}\right)(\theta w_1 + (1-\theta)w_0).$$

Observe also that $\theta w_1 + (1-\theta)w_0 < 0$ for all $\theta \in \Theta$ since $\theta < b$ by assumption. There are two cases:

1. If $C(\pi_{p'}) \geq C(\pi_p)$, then $R_p(\theta) \leq R_{p'}(\theta)$ for all $\theta \in \Theta$, set $D' = D \setminus \{\pi_p\}$ and let the pooling sets be $\{\Theta_{\hat{p}}\}_{\hat{p} \in \mathcal{P} \setminus \{p\}}$. All types which conducted π_p in D now conduct $\pi_{p'}$ in D', and D' is implementable since $Q_p(\theta) = Q_{p'}(\theta)$ for all $\theta \in \Theta$. Observe that pooling is local in D'; all types in $\text{conv}(\Theta_p)$ conduct the

same experiment $\pi_{p'}$.

2. If $C(\pi_{p'}) < C(\pi_p)$, then $R_p(\theta) > R_{p'}(\theta)$ for all $\theta \in \Theta$, set $D' = D \setminus \{\pi_{p'}\}$ and let the pooling sets be $\{\Theta_{\hat{p}}\}_{\hat{p} \in \mathcal{P} \setminus \{p'\}}$. All types which conducted $\pi_{p'}$ in D now conduct π_p in D', and D' is implementable since $Q_p(\theta) = Q_{p'}(\theta)$ for all $\theta \in \Theta$. Observe that pooling is local in D'; all types in $\text{conv}(\Theta_p)$ conduct the same experiment π_p.

In either case, the principal weakly prefers D' to D. Since Θ is finite, we can repeat this construction until there are no further pairs $p, p' \in \mathcal{P}$ such that $\text{conv}(\Theta_p) \cap \text{conv}(\Theta_{p'}) \neq \emptyset$. □

Proof of Lemma 3 (Menus Are Binary Without Loss) Let $D = \{\pi_p\}_{p \in \mathcal{P}} \cup \{\pi_0\}$ be implementable. By Lemma 6 (iii), there exists an implementable menu $\hat{D} = \{\hat{\pi}_p\}_{p \in \mathcal{P}} \cup \{\pi_0\}$ that satisfies (EC-P) such that $E_{\langle \hat{\pi}_p | \beta_p \rangle}[W(\beta)] \geq E_{\langle \pi_p | \beta_p \rangle}[W(\beta)]$ for each $p \in \mathcal{P}$. By Corollary 2, there exist $\{a_k\}_0^K$ such that $a_k > a_{k-1}$ for each k, and for each $p \in \mathcal{P}$, $\Theta_p = (a_{k-1}, a_k] \cap \Theta$ for some k. Hence, it is without loss of generality to let $\mathcal{P} = \{1, \ldots, K\}$ and $\Theta_k = (a_{k-1}, a_k] \cap \Theta$ for each $k \in \mathcal{P}$.

For each $k \in \mathcal{P}$, let π'_k be the unique experiment such that $\text{supp}\langle \pi'_k | \beta_k \rangle = \{\underline{\beta}^k, \overline{\beta}^k\}$ where $\underline{\beta}^k = E_{\langle \hat{\pi}_k | \beta_k \rangle}[\beta | \beta < b]$ and $\overline{\beta}^k = E_{\langle \hat{\pi}_k | \beta_k \rangle}[\beta | \beta \geq b]$. Since U is constant on $[0, b)$ and $[b, 1]$, then we have $E_{\langle \pi'_k | \beta_k \rangle}\left[\left(\frac{\beta - \beta_k}{\beta_k(1-\beta_k)}\right) U(\beta)\right] = E_{\langle \hat{\pi}_k | \beta_k \rangle}\left[\left(\frac{\beta - \beta_k}{\beta_k(1-\beta_k)}\right) U(\beta)\right]$, and for all $\theta \in \Theta$, $E_{\langle \pi'_k | \beta_k \rangle}\left[\left(\frac{\theta}{\beta_k}\beta + \frac{1-\theta}{1-\beta_k}(1-\beta)\right) U(\beta)\right] = E_{\langle \hat{\pi}_k | \beta_k \rangle}\left[\left(\frac{\theta}{\beta_k}\beta + \frac{1-\theta}{1-\beta_k}(1-\beta)\right) U(\beta)\right]$. However, since $G(\cdot | \beta_k)$ is convex, we have $C(\pi'_k) = E_{\langle \pi'_k | \beta_k \rangle}[G(\beta | \beta_k)] < E_{\langle \hat{\pi}_k | \beta_k \rangle}[G(\beta | \beta_k)] = C(\hat{\pi}_k)$ whenever $\hat{\pi}_k \neq \pi'_k$. Next, for each k, fix some $\hat{\theta}_k \in \Theta_k$. Then since $\hat{D}$ satisfies (EC-P), we have

$$
\begin{aligned}
E_{\langle \pi'_k | \beta_k \rangle}\left[\left(\frac{\hat{\theta}_k}{\beta_k}\beta + \frac{1-\hat{\theta}_k}{1-\beta_k}(1-\beta)\right) U(\beta)\right] - C(\pi'_k) &\geq E_{\langle \hat{\pi}_k | \beta_k \rangle}\left[\left(\frac{\hat{\theta}_k}{\beta_k}\beta + \frac{1-\hat{\theta}_k}{1-\beta_k}(1-\beta)\right) U(\beta)\right] - C(\hat{\pi}_k) \\
&= \sum_{\theta_i < \hat{\theta}_k} (\theta_{i+1} - \theta_i) E_{\langle \hat{\pi}_{p^*(\theta_i)} | \beta_{p^*(\theta_i)} \rangle}\left[\left(\frac{\beta - \beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})}\right) U(\beta)\right] \\
&= \sum_{\theta_i < \hat{\theta}_k} (\theta_{i+1} - \theta_i) E_{\langle \pi'_{p^*(\theta_i)} | \beta_{p^*(\theta_i)} \rangle}\left[\left(\frac{\beta - \beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})}\right) U(\beta)\right];
\end{aligned}
$$

this inequality is strict whenever $\hat{\pi}_k \neq \pi'_k$ and binds whenever $\hat{\pi}_k = \pi'_k$. Then for each $k \in \mathcal{P}$, either by Lemma 10 (when $\hat{\pi}_k \neq \pi'_k$) or by letting $\pi''_k = \pi'_k$ (when $\hat{\pi}_k = \pi'_k$), there exists a binary experiment π''_k such that $E_{\langle \pi''_k | \beta_k \rangle}[W(\beta)] \geq E_{\langle \pi'_k | \beta_k \rangle}[W(\beta)]$,

$E_{\langle \pi'_k | \beta_k \rangle}\left[\left(\frac{\beta-\beta_k}{\beta_k(1-\beta_k)}\right)U(\beta)\right] = E_{\langle \pi''_k | \beta_k \rangle}\left[\left(\frac{\beta-\beta_k}{\beta_k(1-\beta_k)}\right)U(\beta)\right]$, and the above constraint binds; i.e.,

$$\begin{aligned} E_{\langle \pi''_k | \beta_k \rangle}&\left[\left(\frac{\hat{\theta}_k}{\beta_k}\beta + \frac{1-\hat{\theta}_k}{1-\beta_k}(1-\beta)\right)U(\beta)\right] - C(\pi''_k) \\ &= \sum_{\theta_i < \hat{\theta}_k} (\theta_{i+1} - \theta_i) E_{\langle \pi'_{p^*(\theta_i)} | \beta_{p^*(\theta_i)} \rangle}\left[\left(\frac{\beta-\beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})}\right)U(\beta)\right] \\ &= \sum_{\theta_i < \hat{\theta}_k} (\theta_{i+1} - \theta_i) E_{\langle \pi''_{p^*(\theta_i)} | \beta_{p^*(\theta_i)} \rangle}\left[\left(\frac{\beta-\beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})}\right)U(\beta)\right] \end{aligned}$$

Let $D' = \{\pi''_k\}_{k \in \mathcal{P}} \cup \{\pi_0\}$. Then for each $k \in \mathcal{P}$, D' satisfies (ECθ-P) for $\theta = \hat{\theta}_k$; we next show that this extends to all $\theta \in \Theta_k$. Observe that since Θ_k is an interval by Lemma 2, we can write

$$\begin{aligned} E_{\langle \pi''_k | \beta_p \rangle}&\left[\left(\frac{\hat{\theta}_k}{\beta_k}\beta + \frac{1-\hat{\theta}_k}{1-\beta_k}(1-\beta)\right)U(\beta)\right] - C(\pi''_k) \\ &= \sum_{\theta_i < \theta} (\theta_{i+1} - \theta_i) E_{\langle \pi''_{p^*(\theta_i)} | \beta_{p^*(\theta_i)} \rangle}\left[\left(\frac{\beta-\beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})}\right)U(\beta)\right] \\ &\quad + (\hat{\theta}_k - \theta) E_{\langle \pi''_k | \beta_k \rangle}\left[\left(\frac{\beta-\beta_k}{\beta_k(1-\beta_k)}\right)U(\beta)\right]. \end{aligned}$$

Re-arranging, this can be written as

$$\begin{aligned} E_{\langle \pi''_k | \beta_k \rangle}&\left[\left(\frac{\hat{\theta}_k}{\beta_k}\beta + \frac{1-\hat{\theta}_k}{1-\beta_k}(1-\beta)\right)U(\beta)\right] - C(\pi''_k) \\ &= \sum_{\theta_i < \theta} (\theta_{i+1} - \theta_i) E_{\langle \pi''_{p^*(\theta_i)} | \beta_{p^*(\theta_i)} \rangle}\left[\left(\frac{\beta-\beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})}\right)U(\beta)\right] \\ &\quad + E_{\langle \pi''_k | \beta_k \rangle}\left[\left(\frac{\hat{\theta}_k}{\beta_k}\beta + \frac{1-\hat{\theta}_k}{1-\beta_k}(1-\beta)\right)U(\beta)\right] - E_{\langle \pi''_k | \beta_k \rangle}\left[\left(\frac{\theta}{\beta_k}\beta + \frac{1-\theta}{1-\beta_k}(1-\beta)\right)U(\beta)\right]. \end{aligned}$$

Re-arranging again yields

$$\begin{aligned} E_{\langle \pi''_k | \beta_k \rangle}&\left[\left(\frac{\theta}{\beta_k}\beta + \frac{1-\theta}{1-\beta_k}(1-\beta)\right)U(\beta)\right] - C(\pi''_k) \\ &= \sum_{\theta_i < \theta} (\theta_{i+1} - \theta_i) E_{\langle \pi''_{p^*(\theta_i)} | \beta_{p^*(\theta_i)} \rangle}\left[\left(\frac{\beta-\beta_{p^*(\theta_i)}}{\beta_{p^*(\theta_i)}(1-\beta_{p^*(\theta_i)})}\right)U(\beta)\right], \end{aligned}$$

and so D' satisfies (EC-P).

Since $\hat{D}$ is implementable, by Lemma 6 (ii), it satisfies (M-P). Hence, since $E_{\langle \hat{\pi}_k | \beta_k \rangle}\left[\left(\frac{\beta-\beta_k}{\beta_k(1-\beta_k)}\right)U(\beta)\right] = E_{\langle \pi''_k | \beta_k \rangle}\left[\left(\frac{\beta-\beta_k}{\beta_k(1-\beta_k)}\right)U(\beta)\right]$ for each $k \in \mathcal{P}$, D' satisfies (M-P). Then by Lemma 6 (i), D' is implementable. □

The Principal's Problem

Proof of Lemma 4 (Exclusion Is At The Low End) Suppose for sake of contradiction that there exist $\theta, \theta' \in \Theta$ with $\theta' > \theta$ such that $\pi^*_{\theta'} \sim_B \pi_0$ but $\pi^*_\theta \not\sim_B \pi_0$. Since D^* solves (COPT), it satisfies (M) and (EC); since it satisfies (EC), it is individually rational. Then since D^* is binary by Theorem 2, Proposition 2 implies that $\eta(\pi^*_{\theta'}) \geq \eta(\pi^*_\theta)$. Since $\pi^*_{\theta'} \sim \pi_0$, $\eta(\pi^*_{\theta'}) = 0$, and hence $\eta(\pi^*_\theta) = 0$. Then, $\pi^*_\theta(\cdot|1) = \pi^*_\theta(\cdot|0)$, and so $\pi^*_\theta \sim \pi_0$, a contradiction. □

Proof of Lemma 5 (Experiments Where Monotonicity Binds) Suppose that $(M(\theta'', \theta'))$ binds for some $\theta'' > \theta'$. By Theorem 2, D^* is binary. Since D^* solves (COPT), it satisfies (M) and (EC); since it satisfies (EC), it is individually rational, and so by Lemma 8(ii), $\frac{\pi^*_\theta(\bar{s}|1)\theta}{\eta(\pi^*_\theta)\theta + \pi^*_\theta(\bar{s}|0)} \geq b$. Then since D^* satisfies (M), by Proposition 2, for each $\theta \in [\theta', \theta'']$, $\eta(\pi^*_{\theta''}) \geq \eta(\pi^*_\theta) \geq \eta(\pi^*_{\theta'})$; by Lemma 8 (iv), $\eta(\pi^*_{\theta''}) = \eta(\pi^*_{\theta'})$, and so the quantities must be equal.

Then since D^* satisfies (EC), Lemma 8 (ii) and Proposition 2 (ii) imply that for each $\theta \in [\theta', \theta'']$,

$$\begin{aligned} E_{\langle \pi^*_\theta | \theta \rangle}[U(\beta)] - C(\pi^*_\theta) &= (\theta \eta(\pi^*_{\theta''}) + \pi^*_\theta(\bar{s}|0))u - C(\pi^*_\theta) = u \sum_{\theta_i < \theta} (\theta_{i+1} - \theta_i)\eta(\pi^*_{\theta_i}) \\ &= (\theta'' \eta(\pi^*_{\theta''}) + \pi^*_{\theta''}(\bar{s}|0))u - C(\pi^*_{\theta''}) - u(\theta'' - \theta)\eta(\pi^*_{\theta''}) \\ \Leftrightarrow \pi^*_\theta(\bar{s}|0)u - C(\pi^*_\theta) &= \pi^*_{\theta''}(\bar{s}|0)u - C(\pi^*_{\theta''}). \end{aligned} \tag{27}$$

Moreover, we also have $\frac{\pi^*_\theta(\bar{s}|1)\theta''}{\theta''\pi^*_\theta(\bar{s}|1) + (1-\theta'')\pi^*_\theta(\bar{s}|0)} \geq \frac{\pi^*_\theta(\bar{s}|1)\theta}{\theta\pi^*_\theta(\bar{s}|1) + (1-\theta)\pi^*_\theta(\bar{s}|0)} \geq b$.

Now suppose that for some $\theta \in [\theta', \theta'')$, $\pi^*_\theta \neq \pi^*_{\theta''}$. We prove a series of claims to arrive at a contradiction.

Claim L5.1: The menu $D' = \{\pi'_t\}_{t \in \Theta} \cup \{\pi_0\}$ formed by letting $\pi'_{\theta''} = \pi^*_\theta$ and $\pi'_t = \pi^*_t$ for all $t \neq \theta''$ satisfies (EC) and (M). (M) follows immediately from the fact that D^* satisfies (M), as does (ECθ) for all types except θ''. Then since $\frac{\pi^*_\theta(\bar{s}|1)\theta''}{\theta''\pi^*_\theta(\bar{s}|1) + (1-\theta'')\pi^*_\theta(\bar{s}|0)} \geq b$, by Lemma 8(ii), we have

$$\begin{aligned} E_{\langle \pi^*_\theta | \theta'' \rangle}[U(\beta)] - C(\pi^*_\theta) &= (\theta'' \pi^*_\theta(\bar{s}|1) + (1 - \theta'')\pi^*_\theta(\bar{s}|0))u - C(\pi^*_\theta) \\ &= (\theta'' \pi^*_{\theta''}(\bar{s}|1) + (1 - \theta'')\pi^*_{\theta''}(\bar{s}|0))u - C(\pi^*_{\theta''}) \text{ (by (27) and since } \eta(\pi^*_{\theta''}) = \eta(\pi^*_\theta)) \\ &= E_{\langle \pi^*_{\theta''} | \theta'' \rangle}[U(\beta)] - C(\pi^*_{\theta''}) \end{aligned}$$

So since D^* satisfies (ECθ) for type θ'', D' must as well. The claim follows.

Claim L5.2: $E_{\langle \pi^*_\theta | \theta'' \rangle}[W(\beta)] < E_{\langle \pi^*_{\theta''} | \theta'' \rangle}[W(\beta)]$**.** Suppose not. Then by construction of D', $\sum_{t\in\Theta} E_{\langle \pi'_t | t \rangle}[W(\beta)]\sigma(t) \geq \sum_{t\in\Theta} E_{\langle \pi^*_t | t\rangle}[W(\beta)]\sigma(t)$. Then since D^* solves COPT, and D' satisfies (EC) and (M) by Claim L5.1, D' solves (COPT) as well — a contradiction, since by Theorem 2, D^* is the unique binary solution to (COPT).

Claim L5.3: $\pi^*_{\theta''}(\bar{s}|0) < \pi^*_\theta(\bar{s}|0)$**.** By Claim L5.2, we have

$$\begin{aligned}
0 &< E_{\langle \pi^*_{\theta''} | \theta'' \rangle}[W(\beta)] - E_{\langle \pi^*_\theta | \theta'' \rangle}[W(\beta)] \\
&= \theta'' \pi^*_{\theta''}(\bar{s}|1) w_1 + (1-\theta'')\pi^*_{\theta''}(\bar{s}|0) w_0 - (\theta'' \pi^*_\theta(\bar{s}|1) w_1 + (1-\theta'')\pi^*_\theta(\bar{s}|0) w_0) \\
&= \theta'' w_1 \eta(\pi^*_{\theta''}) + \pi^*_{\theta''}(\bar{s}|0)(w_1\theta'' + w_0(1-\theta'')) - \left(\theta'' w_1 \eta(\pi^*_\theta) + \pi^*_\theta(\bar{s}|0)(w_1\theta'' + w_0(1-\theta''))\right) \\
&= (\pi^*_{\theta''}(\bar{s}|0) - \pi^*_\theta(\bar{s}|0))(w_1\theta'' + w_0(1-\theta'')) \text{ (since } \eta(\pi^*_{\theta''}) = \eta(\pi^*_\theta)).
\end{aligned}$$

Since $\theta'' < b \equiv \frac{-w_0}{w_1 - w_0}$, the claim follows.

Claim L5.4: $\frac{\pi^*_{\theta''}(\bar{s}|1)\theta}{\theta\pi^*_{\theta''}(\bar{s}|1) + (1-\theta)\pi^*_{\theta''}(\bar{s}|0)} \geq b$**.** By Claim L5.3, we have

$$\frac{\eta(\pi^*_{\theta''})}{\pi^*_{\theta''}(\bar{s}|0)} > \frac{\eta(\pi^*_\theta)}{\pi^*_\theta(\bar{s}|0)} \Rightarrow \frac{1}{1 + \frac{\eta(\pi^*_{\theta''})}{\pi^*_{\theta''}(\bar{s}|0)}} < \frac{1}{1 + \frac{\eta(\pi^*_\theta)}{\pi^*_\theta(\bar{s}|0)}}$$

$$\begin{aligned}
\Rightarrow \frac{\pi^*_{\theta''}(\bar{s}|1)\theta}{\theta\pi^*_{\theta''}(\bar{s}|1) + (1-\theta)\pi^*_{\theta''}(\bar{s}|0)} = \frac{\theta}{\theta + (1-\theta)\frac{1}{1+\frac{\eta(\pi^*_{\theta''})}{\pi^*_{\theta''}(\bar{s}|0)}}} &> \frac{\theta}{\theta + (1-\theta)\frac{1}{1+\frac{\eta(\pi^*_\theta)}{\pi^*_\theta(\bar{s}|0)}}} \\
&= \frac{\pi^*_\theta(\bar{s}|1)\theta}{\theta\pi^*_\theta(\bar{s}|1) + (1-\theta)\pi^*_\theta(\bar{s}|0)} \geq b.
\end{aligned}$$

Claim L5.5: $E_{\langle \pi^*_\theta | \theta \rangle}[W(\beta)] < E_{\langle \pi^*_{\theta''} | \theta \rangle}[W(\beta)]$**.** From Claim L5.3, and since $\theta < b \equiv \frac{-w_0}{w_1 - w_0}$, we have (since $\eta(\pi^*_{\theta''}) = \eta(\pi^*_\theta)$)

$$\begin{aligned}
0 &< (\pi^*_{\theta''}(\bar{s}|0) - \pi^*_\theta(\bar{s}|0))(w_1\theta + w_0(1-\theta)) \\
&= \theta w_1 \eta(\pi^*_{\theta''}) + \pi^*_{\theta''}(\bar{s}|0)(w_1\theta + w_0(1-\theta)) - (\theta w_1 \eta(\pi^*_\theta) + \pi^*_\theta(\bar{s}|0)(w_1\theta + w_0(1-\theta))) \\
&= \theta\pi^*_{\theta''}(\bar{s}|1) w_1 + (1-\theta)\pi^*_{\theta''}(\bar{s}|0) w_0 - (\theta\pi^*_\theta(\bar{s}|1) w_1 + (1-\theta)\pi^*_\theta(\bar{s}|0) w_0) \\
&= E_{\langle \pi^*_{\theta''} | \theta \rangle}[W(\beta)] - E_{\langle \pi^*_\theta | \theta \rangle}[W(\beta)],
\end{aligned}$$

as desired.

Claim L5.6: The menu $D'' = \{\pi''_t\}_{t\in\Theta} \cup \{\pi_0\}$ formed by letting $\pi''_\theta = \pi^*_{\theta''}$ and $\pi''_t = \pi^*_t$ for all $t \neq \theta$ satisfies (EC) and (M). (M) follows immediately from the fact that D^* satisfies (M), as does (ECθ) for all types except θ. Then by Claim L5.4

and Lemma 8(ii), we have

$$\begin{aligned}E_{\langle\pi^*_{\theta''}|\theta\rangle}[U(\beta)] - C(\pi^*_{\theta''}) &= (\theta\pi^*_{\theta''}(\bar{s}|1) + (1-\theta)\pi^*_{\theta''}(\bar{s}|0))u - C(\pi^*_{\theta''}) \\ &= (\theta\pi^*_{\theta}(\bar{s}|1) + (1-\theta)\pi^*_{\theta}(\bar{s}|0))u - C(\pi^*_{\theta}) \text{ (by (27) and since } \eta(\pi^*_{\theta''}) = \eta(\pi^*_{\theta})) \\ &= E_{\langle\pi^*_{\theta''}|\theta''\rangle}[U(\beta)] - C(\pi^*_{\theta''})\end{aligned}$$

So since D^* satisfies (ECθ) for type θ'', D' must as well. The claim follows.

It follows from Claim L5.5 that $\sum_{t\in\Theta} E_{\langle\pi''_t|t\rangle}[W(\beta)]\sigma(t) > \sum_{t\in\Theta} E_{\langle\pi^*_t|t\rangle}[W(\beta)]\sigma(t)$. Then since D'' satisfies (EC) and (M) by Claim L5.6, D^* cannot solve (COPT), a contradiction. ☐

Lemma 15. *Let $D^* = \{\pi^*_\theta\}_{\theta\in\Theta}$ be the unique solution to (COPT) guaranteed by Theorem 2. For each $\theta \in \tilde{\Theta}$, $\max\operatorname{supp}\langle\pi^*_\theta|\theta\rangle > b$.*

Proof. By Theorem 2, D^* is binary; for each $\theta \in \tilde{\Theta}$, label $\operatorname{supp}\langle\pi^*_\theta|\theta\rangle = \{\underline{\beta}_\theta, \overline{\beta}_\theta\}$ with $\underline{\beta}_\theta < \theta < \overline{\beta}_\theta$. By definition, D^* satisfies (EC), and so is individually rational. Then by Lemma 8(ii), $\overline{\beta}_\theta = \frac{\pi^*_\theta(\bar{s}|1)\theta}{\eta(\pi^*_\theta)\theta + \pi^*_\theta(\bar{s}|0)} \geq b$ for each $\theta \in \tilde{\Theta}$.

Now fix $\theta \in \tilde{\Theta}$ and suppose toward a contradiction that $\overline{\beta}_\theta = b$. First, note that we cannot have $\pi^*_{\theta'} = \pi^*_\theta$ for any $\theta' < \theta$: if so, then we have

$$\frac{\overline{\beta}_{\theta'}}{1-\overline{\beta}_{\theta'}} = \frac{\pi^*_\theta(\bar{s}|1)}{\pi^*_\theta(\bar{s}|0)}\frac{\theta'}{1-\theta'} < \frac{\pi^*_\theta(\bar{s}|1)}{\pi^*_\theta(\bar{s}|0)}\frac{\theta}{1-\theta} = \frac{\overline{\beta}_\theta}{1-\overline{\beta}_\theta} = \frac{b}{1-b},$$

and so $\overline{\beta}_{\theta'} < b$, a contradiction since $\pi^*_{\theta'} = \pi^*_\theta$ implies $\theta' \in \tilde{\Theta}$.

Now for each $\theta' < \theta$, let $\pi'_{\theta'} = \pi^*_{\theta'}$; if $\theta = \theta_0$, let $\pi'_\theta = \pi_0$, and if $\theta = \theta_n$ for $n > 0$, let $\pi'_\theta = \pi^*_{\theta_{n-1}}$. Since D^* solves (COPT), it satisfies (M); then if $\theta = \theta_n$ for $n > 0$, it follows from Lemma 5 (since $\pi^*_{\theta_{n-1}} \neq \pi^*_\theta$) and Lemma 8(iv) that

$$\eta(\pi'_{\theta'}) \leq \eta(\pi'_\theta) < \eta(\pi^*_\theta) \leq \eta(\pi^*_{\theta''}) \text{ for each } \theta' < \theta < \theta''. \tag{28}$$

Moreover, if $\theta = \theta_0$, then since $\pi^*_\theta \not\sim_B \pi_0$, we likewise have $\eta(\pi^*_\theta) > \eta(\pi'_\theta)$.

π'_θ satisfies (IRθ): If $\theta = \theta_0$, this is immediate, since then $\pi'_\theta = \pi_0$. If $\theta = \theta_n$ for $n > 0$, then by construction, $\frac{\pi'_\theta(\bar{s}|1)\theta}{\eta(\pi'_\theta)\theta + \pi'_\theta(\bar{s}|0)} > \frac{\pi'_\theta(\bar{s}|1)\theta_{n-1}}{\eta(\pi'_\theta)\theta_{n-1} + \pi'_\theta(\bar{s}|0)} = \frac{\pi^*_{\theta_{n-1}}(\bar{s}|1)\theta_{n-1}}{\eta(\pi^*_{\theta_{n-1}})\theta_{n-1} + \pi^*_{\theta_{n-1}}(\bar{s}|0)} \geq$

b. Then by Lemma 8(ii),

$$\begin{aligned}E_{\langle \pi'_\theta|\theta\rangle}[U(\beta)] - C(\pi'_\theta) &= (\theta\eta(\pi'_\theta) + \pi'_\theta(\bar{s}|0))u - C(\pi'_\theta)\\ &= (\theta_{n-1}\eta(\pi^*_{\theta_{n-1}}) + \pi^*_{\theta_{n-1}}(\bar{s}|0))u - C(\pi^*_{\theta_{n-1}}) + (\theta - \theta_{n-1})\eta(\pi^*_{\theta_{n-1}})\\ &= E_{\langle \pi^*_{\theta_{n-1}}|\theta_{n-1}\rangle}[U(\beta)] - C(\pi^*_{\theta_{n-1}}) \geq 0. \end{aligned} \tag{29}$$

By Theorem 2, D^* is a binary menu; since D^* solves (COPT), it is a screening menu that is individually rational, and satisfies (EC). Then by Proposition 2(ii), for each $\theta' \in \Theta$, $E_{\langle \pi^*_{\theta'}|\theta'\rangle}[U(\beta)] - C(\pi'_{\theta'}) = u\sum_{\theta_i<\theta'}(\theta_{i+1} - \theta_i)\eta(\pi^*_{\theta_i})$. Then for each $\theta' < \theta$, $E_{\langle \pi'_{\theta'}|\theta'\rangle}[U(\beta)] - C(\pi_{\theta'}) = u\sum_{\theta_i<\theta'}(\theta_{i+1} - \theta_i)\eta(\pi'_{\theta_i})$. Moreover, if $\theta = \theta_n$ for $n > 0$, then by (29),

$$\begin{aligned}E_{\langle \pi'_\theta|\theta\rangle}[U(\beta)] - C(\pi'_\theta) &= u\sum_{\theta_i<\theta'}(\theta_{i+1} - \theta_i)\eta(\pi^*_{\theta_i}) + (\theta_n - \theta_{n-1})\eta(\pi'_{\theta_{n-1}})\\ &= u\sum_{\theta_i<\theta}(\theta_{i+1} - \theta_i)\eta(\pi'_{\theta_i});\end{aligned}$$

whereas if $\theta = \theta_0$, then $E_{\langle \pi'_\theta|\theta\rangle}[U(\beta)] - C(\pi'_\theta) = 0$. Finally, since $\eta(\pi^*_\theta) > \eta(\pi'_\theta)$, for each $\theta' > \theta$ we have $E_{\langle \pi^*_{\theta'}|\theta'\rangle}[U(\beta)] - C(\pi_{\theta'}) > u\sum_{\theta_i\leq\theta}(\theta_{i+1} - \theta_i)\eta(\pi'_{\theta_i}) + u\sum_{\theta_i\in(\theta,\theta')}(\theta_{i+1} - \theta_i)\eta(\pi^*_{\theta_i})$. Then by Lemma 10 and Lemma 8(iv), for each $\theta' > \theta$, we can construct a binary $\pi'_{\theta'}$ such that (a) $E_{\langle \pi'_{\theta'}|\theta'\rangle}[W(\beta)] > E_{\langle \pi^*_{\theta'}|\theta'\rangle}[W(\beta)]$, (b) $\eta(\pi'_{\theta'}) = \eta(\pi^*_{\theta'})$, and (c) $E_{\langle \pi'_{\theta'}|\theta'\rangle}[U(\beta)] - C(\pi_{\theta'}) = u\sum_{\theta_i<\theta}(\theta_{i+1} - \theta_i)\eta(\pi'_{\theta_i}) + u\sum_{\theta_i\in(\theta,\theta')}(\theta_{i+1} - \theta_i)\eta(\pi^*_{\theta_i}) = u\sum_{\theta_i<\theta'}(\theta_{i+1} - \theta_i)\eta(\pi'_{\theta_i}) \geq 0$, and hence such that $\pi'_{\theta'}$ satisfies (IRθ').

Now consider the menu $D' = \{\pi'_t\}_{t\in\Theta} \cup \{\pi_0\}$. Since we have shown that $\pi'_{\theta'}$ is binary and satisfies (IRθ') for each $\theta' \leq \theta$, and since D^* is binary and individually rational and $\pi'_{\theta'} = \pi^*_{\theta'}$ for each $\theta' < \theta$, D' is individually rational. Then since $E_{\langle \pi'_{\theta'}|\theta'\rangle}[U(\beta)] - C(\pi_{\theta'}) = u\sum_{\theta_i<\theta'}(\theta_{i+1} - \theta_i)\eta(\pi'_{\theta_i})$ for all $\theta' \in \Theta$, by Proposition 2(ii), D' satisfies (EC). Since D^* is a screening menu that is individually rational and satisfies (M) and (EC), by Proposition 2(i), $\eta(\pi'_{\theta''}) \geq \eta(\pi'_{\theta'})$ for each $\theta'' > \theta'$ with $\theta', \theta'' \neq \theta$; by (28) when $\theta > \theta_0$, and since $\eta(\pi'_\theta) = 0$ when $\theta = \theta_0$, this holds for all θ', θ''. Then by Proposition 2(i), D' satisfies (M).

Finally, note that since W is affine on $[0, b]$, and by assumption, $\text{supp}\langle \pi^*_\theta|\theta\rangle \subseteq [0, b]$, $E_{\langle \pi^*_\theta|\theta\rangle}[W(\beta)] = 0 \leq E_{\langle \pi'_\theta|\theta\rangle}[W(\beta)]$. Then since $\pi'_{\theta'} = \pi^*_{\theta'}$ for each $\theta' < \theta$, and $E_{\langle \pi'_{\theta'}|\theta'\rangle}[W(\beta)] > E_{\langle \pi^*_{\theta'}|\theta'\rangle}[W(\beta)]$ for each $\theta' > \theta$, we have $\sum_{\theta'\in\Theta}\sigma(\theta')E_{\langle \pi^*_{\theta'}|\theta'\rangle}[W(\beta)] \geq$

$\sum_{\theta' \in \Theta} \sigma(\theta') E_{\langle \pi_0 | \theta' \rangle}[W(\beta)]$ (where this inequality is strict if $\theta < \theta_N$). Then D' also solves (COPT), a contradiction by Theorem 2. □

Lemma 16 (Positive Lagrange Multipliers). *Let $\{\lambda_n^*\}_{n=0}^N, \{\delta_n^*\}_{n=0}^N$ be as in Theorem 2. If $\theta_n \geq \underline{\theta}$, $\lambda_n^* > 0$.*

Proof. By Lemma 14, $\lambda_n^* \geq 0$. Suppose, for sake of contradiction, that $\lambda_n^* = 0$. Then by (TBTθ_n), if $n < N$,

$$\pi_{\theta_n}^* \in \arg\max_{\pi} E_{\langle \pi | \theta_n \rangle} \left[W(\beta)\sigma(\theta_n) - \left(\delta_{n+1}^* - \delta_n^* \mathbf{1}_{n>0} + (\theta_{n+1} - \theta_n) \textstyle\sum_{y=n+1}^N \lambda_y^* \right) \left(\frac{\beta - \theta_n}{\theta_n(1-\theta_n)} \right) U(\beta) \right]$$

and if $n = N$,

$$\pi_{\theta_n}^* \in \arg\max_{\pi} E_{\langle \pi | \theta_n \rangle} \left[W(\beta)\sigma(\theta_n) + \delta_n^* \left(\frac{\beta - \theta_n}{\theta_n(1-\theta_n)} \right) U(\beta) \right].$$

In either case, the objective function is constant for $\beta < b$ and piecewise linear in β with a jump discontinuity at $\beta = b$. This leaves three possibilities: (i) $\text{supp}\langle \pi_{\theta_n}^* | \theta_n \rangle = \{\theta_n\}$, (ii) $\text{supp}\langle \pi_{\theta_n}^* | \theta_n \rangle = \{0, b\}$, or (iii) $\text{supp}\langle \pi_{\theta_n}^* | \theta_n \rangle = \{0, 1\}$. Case (i) cannot hold, since $\theta_n \geq \underline{\theta}$. If case (ii) or case (iii) holds, $E_{\langle \pi_{\theta_n}^* | \theta_n \rangle}[U(\beta) - G(\beta | \theta_n)] = -\infty$. Since $\pi_{\theta_n}^*$ is the experiment that type θ_n conducts in the unique optimal menu D^*, by Theorem 2, π_{θ_n} must satisfy (IRθ). Hence, we reach a contradiction. □

Lemma 17. *The "distortion term" $R(b, \theta_n)/\lambda_n^*$ is strictly positive whenever $\underline{\theta} \leq \theta_n < \theta_N$ and either $n = 0$ or $\delta_n^* = 0$.*

Proof. Suppose that $\underline{\theta} \leq \theta_n < \theta_N$ and either $n = 0$ or $\delta_n^* = 0$. Then we have

$$\begin{aligned} R(b, \theta_n)/\lambda_n^* &= \left(\frac{b - \theta_n}{\theta_n(1-\theta_n)} \right) u \left(\left(\delta_{n+1}^* + (\theta_{n+1} - \theta_n) \sum_{i=n+1}^N \lambda_i^* \right) \mathbf{1}_{n<N} - \delta_n^* \mathbf{1}_{n>0} \right) / \lambda_n^* \\ &= \left(\frac{b - \theta_n}{\theta_n(1-\theta_n)} \right) u \left(\delta_{n+1}^* + (\theta_{n+1} - \theta_n) \sum_{i=n+1}^N \lambda_i^* \right) / \lambda_n^* > 0, \end{aligned}$$

since $\theta_n < b$; $\delta_{n+1}^* \geq 0$; and by Lemma 16, $\lambda_i^* > 0$ for all i with $\theta_i > \underline{\theta}$. □

The Pareto Frontier and Distortion

Lemma 18 (Pareto Improvements Are Not Less Informative). *If π is on the Pareto frontier for type $\theta \in \tilde{\Theta}$, and is a Pareto improvement on π_θ^* for type θ, then $\pi_\theta^* \not\succeq_B \pi$.*

Proof. By Corollary 1, $|\operatorname{supp}\langle\pi_\theta|\theta\rangle| = 2$. Write $\operatorname{supp}\langle\pi_\theta|\theta\rangle = \{\underline{\beta}, \overline{\beta}\}$ and $\operatorname{supp}\langle\pi^*_\theta|\theta\rangle = \{\underline{\beta}^*, \overline{\beta}^*\}$ for $\underline{\beta} < \theta < \overline{\beta}$ and $\underline{\beta}^* < \theta < \overline{\beta}^*$. By Lemma 15, since $\theta \in \tilde{\Theta}$, we must have $\overline{\beta}^* > b$.

Suppose toward a contradiction that π^*_θ is Blackwell-more informative than π. Then $\langle\pi^*_\theta|\theta\rangle$ is a mean-preserving spread of $\langle\pi|\theta\rangle$. If $\overline{\beta} < b$, then (IRθ) fails, a contradiction by Theorem 2. If $\overline{\beta} = b$, then since $\overline{\beta}^* > b$, it is immediate that $E_{\langle\pi_\theta|\theta\rangle}[W(\beta)] = 0 < E_{\langle\pi^*_\theta|\theta\rangle}[W(\beta)]$, a contradiction. Finally, if $\overline{\beta} > b$, then since W is convex and strictly convex about the threshold, it follows from Lemma 13 that $E_{\langle\pi|\theta\rangle}[W(\beta)] < E_{\langle\pi^*_\theta|\theta\rangle}[W(\beta)]$, a contradiction. □

Lemma 19 (Monotonicity of Pareto Dominance). *If π' Pareto dominates π for type θ, $\operatorname{supp}\langle\pi|\theta\rangle = \{\underline{\beta}, \overline{\beta}\}$ and $\operatorname{supp}\langle\pi'|\theta\rangle = \{\underline{\beta}', \overline{\beta}'\}$ with $\overline{\beta}, \overline{\beta}' \geq b$, and $\pi'(\underline{s}|1) < \pi(\underline{s}|1)$, then π' Pareto dominates π for all $\theta' > \theta$.*

Proof. For each $\theta' > \theta$, label $\operatorname{supp}\langle\pi|\theta'\rangle = \{\underline{\beta}_{\theta'}, \overline{\beta}_{\theta'}\}$ and $\operatorname{supp}\langle\pi'|\theta\rangle = \{\underline{\beta}'_{\theta'}, \overline{\beta}'_{\theta'}\}$ with $\underline{\beta}_{\theta'} < \theta' < \overline{\beta}_{\theta'}$ and $\underline{\beta}'_{\theta'} < \theta' < \overline{\beta}'_{\theta'}$.

Then since $\overline{\beta}, \overline{\beta}' \geq b$, for each $\theta' > \theta$,

$$\frac{\overline{\beta}_{\theta'}}{1-\overline{\beta}_{\theta'}} = \frac{\pi(\overline{s}|1)}{\pi(\overline{s}|0)}\frac{\theta'}{1-\theta'} > \frac{\pi(\overline{s}|1)}{\pi(\overline{s}|0)}\frac{\theta}{1-\theta} = \frac{\overline{\beta}}{1-\overline{\beta}} \geq \frac{b}{1-b}, \text{ and}$$
$$\frac{\overline{\beta}'_{\theta'}}{1-\overline{\beta}'_{\theta'}} = \frac{\pi'(\overline{s}|1)}{\pi'(\overline{s}|0)}\frac{\theta'}{1-\theta'} > \frac{\pi'(\overline{s}|1)}{\pi'(\overline{s}|0)}\frac{\theta}{1-\theta} = \frac{\overline{\beta}'}{1-\overline{\beta}'} \geq \frac{b}{1-b}.$$

Hence, $\overline{\beta}_{\theta'}, \overline{\beta}'_{\theta'} > b$.

Moreover, since $\overline{\beta}, \overline{\beta}' \geq b$, $E_{\langle\pi|\theta\rangle}[W(\beta)] = \pi(\overline{s}|0)w_0 + \theta(\pi(\overline{s}|1)w_1 - \pi(\overline{s}|0)w_0) \geq 0$ and $E_{\langle\pi'|\theta\rangle}[W(\beta)] = \pi'(\overline{s}|0)w_0 + \theta(\pi'(\overline{s}|1)w_1 - \pi'(\overline{s}|0)w_0) \geq 0$.

Now suppose for sake of contradiction that the principal weakly prefers π to π' when his prior is θ'. Then

$$\begin{aligned} 0 < E_{\langle\pi'|\theta'\rangle}[W(\beta)] &= \pi'(\overline{s}|0)w_0 + \theta'(\pi'(\overline{s}|1)w_1 - \pi'(\overline{s}|0)w_0) \\ &\leq E_{\langle\pi|\theta'\rangle}[W(\beta)] = \max\{0, \pi(\overline{s}|0)w_0 + \theta'(\pi(\overline{s}|1)w_1 - \pi(\overline{s}|0)w_0)\} \\ \Rightarrow &= \pi(\overline{s}|0)w_0 + \theta'(\pi(\overline{s}|1)w_1 - \pi(\overline{s}|0)w_0), \end{aligned}$$

but since π' Pareto dominates π for type θ,

$$\begin{aligned}E_{\langle\pi'|\theta\rangle}[W(\beta)] &= \pi'(\bar{s}|0)w_0 + \theta(\pi'(\bar{s}|1)w_1 - \pi'(\bar{s}|0)w_0) \\ &\geq E_{\langle\pi|\theta\rangle}[W(\beta)] = \max\{0, \pi(\bar{s}|0)w_0 + \theta(\pi(\bar{s}|1)w_1 - \pi(\bar{s}|0)w_0)\} \\ &\geq \pi(\bar{s}|0)w_0 + \theta(\pi(\bar{s}|1)w_1 - \pi(\bar{s}|0)w_0).\end{aligned}$$

Adding these inequalities yields

$$\begin{aligned}\begin{array}{l}\theta(\pi(\bar{s}|1)w_1 - \pi(\bar{s}|0)w_0) \\ +\theta'(\pi'(\bar{s}|1)w_1 - \pi'(\bar{s}|0)w_0)\end{array} &\leq \begin{array}{l}\theta'(\pi(\bar{s}|1)w_1 - \pi(\bar{s}|0)w_0) \\ +\theta(\pi'(\bar{s}|1)w_1 - \pi'(\bar{s}|0)w_0)\end{array} \\ \implies (\theta' - \theta)(\pi'(\bar{s}|1)w_1 - \pi'(\bar{s}|0)w_0) &\leq (\theta' - \theta)(\pi(\bar{s}|1)w_1 - \pi(\bar{s}|0)w_0) \\ \implies (\pi'(\bar{s}|1)w_1 - \pi'(\bar{s}|0)w_0) &\leq (\pi(\bar{s}|1)w_1 - \pi(\bar{s}|0)w_0) \\ \implies (\pi'(\bar{s}|1) - \pi(\bar{s}|1))w_1 &\leq (\pi'(\bar{s}|0) - \pi(\bar{s}|0))w_0\end{aligned}$$

Then since $\pi'(\bar{s}|1) > \pi(\bar{s}|1)$, we have $\pi'(\bar{s}|0) < \pi(\bar{s}|0)$ since $w_1 > 0$ and $w_0 < 0$. It follows that

$$\begin{aligned}E_{\langle\pi'|\theta'\rangle}[W(\beta)] = \pi'(\bar{s}|0)w_0(1-\theta') + \theta'\pi'(\bar{s}|1)w_1 &> \pi(\bar{s}|0)w_0(1-\theta') + \theta'\pi(\bar{s}|1)w_1 \\ &= E_{\langle\pi'|\theta\rangle}[W(\beta)],\end{aligned}$$

a contradiction. Hence, the principal strictly prefers π' to π at θ'.

Now suppose toward a contradiction that π' does not give a type-θ' agent a strictly greater expected payoff than π. Then since $\overline{\beta}_{\theta'}, \overline{\beta}'_{\theta'} > b$, it follows from Lemma 8(ii) that

$$\begin{aligned}E_{\langle\pi'|\theta'\rangle}[U(\beta)] - C(\pi') &= (\pi'(\bar{s}|0) + \theta'\eta(\pi'))u - C(\pi') \\ &\leq E_{\langle\pi|\theta'\rangle}[U(\beta)] - C(\pi) = (\pi(\bar{s}|0) + \theta'\eta(\pi))u - C(\pi)\end{aligned}$$

but

$$\begin{aligned}E_{\langle\pi'|\theta\rangle}[U(\beta)] - C(\pi') &= (\pi'(\bar{s}|0) + \theta\eta(\pi'))u - C(\pi') \\ &\geq E_{\langle\pi|\theta\rangle}[U(\beta)] - C(\pi) = (\pi(\bar{s}|0) + \theta\eta(\pi))u - C(\pi).\end{aligned}$$

Adding these inequalities yields

$$\begin{aligned} \theta\eta(\pi)u + \theta'\eta(\pi')u &\leq \theta\eta(\pi')u + \theta'\eta(\pi)u \\ \implies (\theta' - \theta)\eta(\pi')u &\leq (\theta' - \theta)\eta(\pi)u \\ \implies \pi'(\bar{s}|1) - \pi'(\bar{s}|0) = \eta(\pi') &\leq \eta(\pi) = \pi(\bar{s}|1) - \pi(\bar{s}|0). \end{aligned}$$

Then since $\pi'(\bar{s}|1) > \pi(\bar{s}|1)$, we have $\pi'(\bar{s}|0) > \pi(\bar{s}|0)$. Since the principal strictly prefers π' to π at θ, then

$$\begin{aligned} &\theta\pi'(\bar{s}|1)w_1 + (1-\theta)\pi'(\bar{s}|0)w_0 > \theta\pi(\bar{s}|1)w_1 + (1-\theta)\pi(\bar{s}|0)w_0 \\ \implies &(1-\theta)w_0(\pi'(\bar{s}|0) - \pi(\bar{s}|0)) > \theta w_1(\pi(\bar{s}|1) - \pi'(\bar{s}|1)) > \theta w_1(\pi(\bar{s}|0) - \pi'(\bar{s}|0)) \\ &\implies -(1-\theta)w_0 > \theta w_1 \text{ (since } \pi'(\bar{s}|0) > \pi(\bar{s}|0)) \\ &\implies \theta w_1 + (1-\theta)w_0 > 0 = bw_1 + (1-b)w_0 \\ &\implies \theta > b, \end{aligned}$$

a contradiction. It follows that π' is a Pareto improvement on π for θ'. □

Proof of Proposition 4 (No Distortion at the Top) There are two possible cases.

Case I: $\pi^*_{\theta_N} \neq \pi^*_{\theta_{N-1}}$. By Theorem 2, $\pi^*_{\theta_N}$ solves (TBTθ_n) for $\theta = \theta_N$. If $\pi^*_{\theta_N} \neq \pi^*_{\theta_{N-1}}$, then by Lemma 5, (M) does not bind for type θ_N. Then by (CS), $\delta^*_N = 0$, and so $\pi^*_{\theta_N}$ solves

$$\max_{\pi \in \Pi} E_{\langle \pi|\theta_N \rangle} \left[\frac{\sigma(\theta_N)}{\lambda^*_{\theta_N}} W(\beta) + U(\beta) - G(\beta|\theta_N) \right].$$

It follows that $\langle \pi^*_{\theta_N} | \theta_N \rangle$ solves (SPPθ) for $\lambda_p = \frac{\sigma(\theta_N)}{\lambda^*_{\theta_N}}$ and $\lambda_a = 1$. The claim follows from Proposition 3.

Case II: $\pi^*_{\theta_N} = \pi^*_{\theta_{N-1}}$. If $\pi^*_{\theta_N} = \pi^*_{\theta_{N-1}}$, suppose for sake of contradiction that there exists some π' which Pareto improves upon $\pi^*_{\theta_N}$ for type θ_N. Since D^* satisfies (EC), $\pi^*_{\theta_N}$ satisfies (IRθ) for $\theta = \theta_N$; then since $E_{\langle \pi_0|\theta_N \rangle}[U(\beta)] - C(\pi_0) = 0 = E_{\langle \pi_0|\theta_N \rangle}[W(\beta)]$, $\pi' \not\sim_B \pi_0$. Without loss, let π' be on the Pareto frontier for type θ_N; then by Corollary 1, π' is binary. Observe that $D' = \{\pi'\} \cup D^* \setminus \{\pi_{\theta^*_N}\}$ gives the principal and the type θ_N agent a weakly higher payoff (with one inequality strict). We consider two subcases.

Case II.1: $\eta(\pi') \geq \eta(\pi^*_{\theta_N})$. Since π' is a Pareto improvement on $\pi^*_{\theta_N}$, we have

$$\begin{aligned} E_{\langle \pi' | \theta_N \rangle}[U(\beta)] - C(\pi') &\geq E_{\langle \pi^*_{\theta_N} | \theta_N \rangle}[U(\beta)] - C(\pi^*_{\theta_N}) \\ &= \sum_{i<N} (\theta_{i+1} - \theta_i) E_{\langle \pi_{\theta_i} | \theta_i \rangle} \left[\left(\frac{\beta - \theta_i}{\theta_i(1-\theta_i)} \right) U(\beta) \right]. \end{aligned} \tag{30}$$

since D^* solves (COPT). If (30) holds with equality, then by Proposition 1, D' is implementable; moreover, since π' is a Pareto improvment on $\pi^*_{\theta_N}$, $E_{\langle \pi' | \theta_N \rangle}[W(\beta)] > E_{\langle \pi^*_{\theta_N} | \theta_N \rangle}[W(\beta)]$, and so D' attains a higher value than D^* in (COPT), a contradiction. Alternatively, if (30) is strict, then by Lemma 10, we can construct an experiment π'' such that $E_{\langle \pi'' | \theta_N \rangle}[W(\beta)] > E_{\langle \pi' | \theta_N \rangle}[W(\beta)] \geq E_{\langle \pi^*_{\theta_N} | \theta_N \rangle}[W(\beta)]$, $\eta(\pi'') = \eta(\pi') \geq \eta(\pi^*_{\theta_N})$, and (30) holds with equality. It follows that $D'' = \{\pi''\} \cup D^* \setminus \{\pi^*_{\theta_N}\}$ is implementable and attains a higher value than D^* in (COPT), a contradiction.

Case II.2: $\eta(\pi') < \eta(\pi^*_{\theta_N})$. Then $\pi^*_{\theta_N} \not\succsim_B \pi_0$, since $\eta(\pi^*_{\theta_N}) > 0$. Since π' is a Pareto improvement on $\pi^*_{\theta_N}$, we have $E_{\langle \pi' | \theta_N \rangle}[W(\beta)] \geq E_{\langle \pi^*_{\theta_N} | \theta_N \rangle}[W(\beta)]$, or equivalently,

$$\pi'(\bar{s}|0)(\theta_N w_1 + (1-\theta_N)w_0) + \theta_N \eta(\pi') w_1 \geq \pi^*_{\theta_N}(\bar{s}|0)(\theta_N w_1 + (1-\theta_N)w_0) + \theta_N \eta(\pi^*_{\theta_N}) w_1.$$

Since $\eta(\pi') < \eta(\pi^*_{\theta_N})$, the above inequality implies that $\pi'(\bar{s}|0) < \pi^*_{\theta_N}(\bar{s}|0)$ since $\theta_N w_1 + (1-\theta_N)w_0 < 0$ (which follows from the assumption that $\theta_N < b$). We prove two intermediate claims.

Claim P4.1: In Case II.2, for any $\theta < \theta^*_N$, π' is a Pareto improvement on $\pi^*_{\theta_N}$ for type θ. To see this, note that

$$\begin{aligned} (\pi'(\bar{s}|0) - &\pi^*_{\theta_N}(\bar{s}|0))(\theta w_1 + (1-\theta)w_0) + \theta(\eta(\pi') - \eta(\pi^*_{\theta_N}))w_1 \\ &\geq (\pi'(\bar{s}|0) - \pi^*_{\theta_N}(\bar{s}|0))(\theta_N w_1 + (1-\theta_N)w_0) + \theta_N(\eta(\pi') - \eta(\pi^*_{\theta_N}))w_1 \\ &\geq 0, \end{aligned}$$

where the first inequality follows from the fact that $\theta w_1 + (1-\theta)w_0 < \theta_N w_1 + (1-\theta_N)w_0$, $\pi'(\bar{s}|0) - \pi^*_{\theta_N}(\bar{s}|0) < 0$, and $\eta(\pi') < \eta(\pi^*_{\theta_N})$. The second inequality follows from the fact that $E_{\langle \pi' | \theta_N \rangle}[W(\beta)] \geq E_{\langle \pi^*_{\theta_N} | \theta_N \rangle}[W(\beta)]$. Therefore

$$\pi'(\bar{s}|0)(\theta w_1 + (1-\theta)w_0) + \theta\eta(\pi')w_1 \geq \pi^*_{\theta_N}(\bar{s}|0)(\theta w_1 + (1-\theta)w_0) + \theta\eta(\pi^*_{\theta_N})w_1;$$
$$\Rightarrow E_{\langle \pi' | \theta \rangle}[W(\beta)] = \max\{0, \pi'(\bar{s}|0)(\theta w_1 + (1-\theta)w_0) + \theta\eta(\pi')w_1\}$$
$$\geq \max\{0, \pi^*_{\theta_N}(\bar{s}|0)(\theta w_1 + (1-\theta)w_0) + \theta\eta(\pi^*_{\theta_N})w_1\} = E_{\langle \pi^*_{\theta_N} | \theta_N \rangle}[W(\beta)].$$

For each $\theta \in \Theta$, label $\text{supp}\langle \pi' | \theta \rangle = \{\underline{\beta}'_\theta, \overline{\beta}'_\theta\}$ and $\text{supp}\langle \pi^*_{\theta_N} | \theta \rangle = \{\underline{\beta}^*_\theta, \overline{\beta}^*_\theta\}$ with $\underline{\beta}'_\theta < \overline{\beta}'_\theta$ and $\underline{\beta}^*_\theta < \overline{\beta}^*_\theta$. Since $\pi(\bar{s}|0)(\theta w_1 + (1-\theta)w_0) + \theta\eta(\pi)w_1 \geq 0 \Leftrightarrow \text{supp}\langle \pi | \theta \rangle \cap [b, 1] \neq \emptyset$ for any binary π, it follows that for any $\theta \in \Theta$, $\overline{\beta}^*_\theta \geq b \implies \overline{\beta}'_\theta \geq b$. Next, observe that

$$\begin{aligned}
&((\pi'(\bar{s}|0) - \pi^*_{\theta_N}(\bar{s}|0) + \theta(\eta(\pi') - \eta(\pi^*_{\theta_N})))u - C(\pi') + C(\pi^*_{\theta_N}) \\
&\qquad \geq ((\pi'(\bar{s}|0) - \pi^*_{\theta_N}(\bar{s}|0) + \theta_N(\eta(\pi') - \eta(\pi^*_{\theta_N})))u - C(\pi') + C(\pi^*_{\theta_N}) \\
&\qquad \geq 0,
\end{aligned}$$

where the first inequality follows from the fact that $\eta(\pi') < \eta(\pi^*_{\theta_N})$ and the second follows from Lemma 8(ii) and the facts that $E_{\langle \pi' | \theta_N \rangle}[U(\beta)] - C(\pi') \geq E_{\langle \pi^*_{\theta_N} | \theta_N \rangle}[U(\beta)] - C(\pi^*_{\theta_N}) \geq 0$ and $\pi', \pi^*_{\theta_N} \not\succ_B \pi_0$. Moreover, since $\eta(\pi') < \eta(\pi^*_{\theta_N})$ and $\pi'(\bar{s}|0) < \pi^*_{\theta_N}(\bar{s}|0)$, we must have $C(\pi') < C(\pi^*_{\theta_N})$. Therefore,

$$(\pi'(\bar{s}|0) + \theta\eta(\pi'))u - C(\pi') \geq (\pi^*_{\theta_N}(\bar{s}|0) + \theta\eta(\pi^*_{\theta_N}))u - C(\pi^*_{\theta_N}).$$

and since $\overline{\beta}^*_\theta \geq b \implies \overline{\beta}'_\theta \geq b$, by Lemma 8(ii),

$$\begin{aligned}
E_{\langle \pi' | \theta \rangle}[U(\beta)] - C(\pi') &= \begin{cases} (\pi'(\bar{s}|0) + \theta\eta(\pi'))u - C(\pi'), & \overline{\beta}'_\theta \geq b, \\ -C(\pi'), & \overline{\beta}'_\theta < b; \end{cases} \\
&\geq \begin{cases} (\pi^*_{\theta_N}(\bar{s}|0) + \theta\eta(\pi^*_{\theta_N}))u - C(\pi^*_{\theta_N}), & \overline{\beta}^*_\theta \geq b, \\ -C(\pi^*_{\theta_N}), & \overline{\beta}^*_\theta < b; \end{cases} \\
&= E_{\langle \pi^*_{\theta_N} | \theta \rangle}[U(\beta)] - C(\pi^*_{\theta_N}).
\end{aligned}$$

Thus, π' Pareto improves upon $\pi^*_{\theta_N}$ for all $\theta \leq \theta_N$. ■

Claim P4.2: Let $z = \min\{n | \pi^*_{\theta_n} = \pi^*_{\theta_N}\}$**. In Case II.2, there exists a binary** $\hat{\pi} \in \Pi$ **with** $\eta(\hat{\pi}) > \eta(\pi^*_{\theta_{z-1}})$**, and thus** $\hat{\pi} \not\succ_B \pi_0$**, that is a Pareto improvement on** $\pi^*_{\theta_N}$ **for all** $\theta \leq \theta_N$**.** If $\eta(\pi') \geq \eta(\pi^*_{\theta_{z-1}})$ then by Claim P4.1 we can let $\hat{\pi} = \pi'$. Suppose not and $\eta(\pi') < \eta(\pi^*_{\theta_{z-1}})$. First note that $\eta(\pi^*_{\theta_N}) > \eta(\pi^*_{\theta_{z-1}})$ by Lemma 5. Then there exists $\alpha \in (0,1)$ such that $\alpha\eta(\pi^*_{\theta_N}) + (1-\alpha)\eta(\pi') > \eta(\pi^*_{\theta_{z-1}})$. Then, observe that the experiment $\hat{\pi} = \alpha\pi + (1-\alpha)\pi'$ Pareto dominates $\pi^*_{\theta_N}$ for any $\theta \leq \theta_N$ since

$$E_{\langle\hat{\pi}|\theta\rangle}[W(\beta)] = \alpha E_{\langle\pi'|\theta\rangle}[W(\beta)] + (1-\alpha)E_{\langle\pi^*_{\theta_N}|\theta\rangle}[W(\beta)] \geq E_{\langle\pi^*_{\theta_N}|\theta\rangle}[W(\beta)]$$

and

$$\begin{aligned} E_{\langle\hat{\pi}|\theta\rangle}[U(\beta)] - C(\hat{\pi}) &= \alpha(E_{\langle\pi'|\theta\rangle}[U(\beta)] - C(\pi')] + (1-\alpha)(E_{\langle\pi^*_{\theta_N}|\theta\rangle}[U(\beta)] - C(\pi^*_{\theta_N})) \\ &\geq E_{\langle\pi^*_{\theta_N}|\theta\rangle}[U(\beta)] - C(\pi^*_{\theta_N}) \end{aligned}$$

where the first equality follows from linearity of $C(\pi)$ (Axioms 2 and 3 in Pomatto et al. (2023)). Therefore, $\hat{\pi}$ Pareto dominates $\pi^*_{\theta_N}$ for all $\theta \leq \theta_N$. Moreover, observe that $\eta(\hat{\pi}) = \hat{\pi}(\bar{s}|1) - \hat{\pi}(\bar{s}|0) = \alpha\eta(\pi^*_{\theta_N}) + (1-\alpha)\eta(\pi') > \eta(\pi^*_{\theta_N})$. The claim follows. ∎

Let $\hat{\pi}$ be the experiment from Claim P4.2. Then we have

$$\begin{aligned} E_{\langle\hat{\pi}|\theta_z\rangle}[U(\beta)] - C(\hat{\pi}) &\geq E_{\langle\pi^*_z|\theta_z\rangle}[U(\beta)] - C(\pi^*_{\theta_N}) \\ &= \sum_{i<z}(\theta_{i+1} - \theta_i)E_{\langle\pi_{\theta_i}|\theta_i\rangle}\left[\left(\frac{\beta - \theta_i}{\theta_i(1-\theta_i)}\right)U(\beta)\right] \end{aligned} \tag{31}$$

since $\hat{\pi}$ Pareto dominates $\pi^*_{\theta_N}$ for θ_z. Then since $\hat{\pi} \not\succ_B \pi_0$, by Lemma 8(ii), we must have $\frac{\hat{\pi}(\bar{s}|1)\theta_z}{\eta(\hat{\pi})\theta_z + \hat{\pi}(\bar{s}|0} \geq b$, and hence $\frac{\hat{\pi}(\bar{s}|1)\theta_n}{\eta(\hat{\pi})\theta_n + \hat{\pi}(\bar{s}|0} \geq b$ for each $n \geq z$.

If (31) holds with equality, then by Lemma 8(ii),

$$\begin{aligned} &(\hat{\pi}(\bar{s}|0) + \theta_z\eta(\hat{\pi}))u - C(\hat{\pi}) = u\sum_{i<z}(\theta_{i+1} - \theta_i)\eta(\pi^*_{\theta_i}); \\ \Rightarrow\ &(\hat{\pi}(\bar{s}|0) + \theta_n\eta(\hat{\pi}))u - C(\hat{\pi}) = u\sum_{i<z}(\theta_{i+1} - \theta_i)\eta(\pi^*_{\theta_i}) + \sum_{i=z}^{n-1}(\theta_{i+1} - \theta_i)\eta(\hat{\pi})\forall n \geq z, \end{aligned}$$

and so $D' = \{\hat{\pi}\}_{\theta \geq \theta_z} \cup D^* \setminus \{\pi^*_\theta\}_{\theta \geq \theta_z}$ must be individually rational by Lemma 8(ii), and thus satisfies (EC) and ((M)) by Proposition 2. Moreover, since $\hat{\pi}$ is a Pareto improvment on $\pi^*_{\theta_N}$, $E_{\langle\hat{\pi}|\theta_N\rangle}[W(\beta)] > E_{\langle\pi^*_{\theta_N}|\theta_N\rangle}[W(\beta)]$, and so D' attains a higher value than D^* in (COPT), a contradiction.

Alternatively, if (31) is strict, then by Lemma 8(ii),

$$(\hat{\pi}(\bar{s}|0) + \theta_z \eta(\hat{\pi}))u - C(\hat{\pi}) > u \sum_{i<z} (\theta_{i+1} - \theta_i)\eta(\pi^*_{\theta_i});$$

$$\Rightarrow E_{\langle \hat{\pi}|\theta_n \rangle}[U(\beta)] - C(\hat{\pi}) = (\hat{\pi}(\bar{s}|0) + \theta_n \eta(\hat{\pi}))u - C(\hat{\pi}) \tag{32}$$

$$> u \sum_{i<z} (\theta_{i+1} - \theta_i)\eta(\pi^*_{\theta_i}) + \sum_{i=z}^{n-1} (\theta_{i+1} - \theta_i)\eta(\hat{\pi}) \forall n \geq z. \tag{33}$$

Then by Lemma 10 and Lemma 8, for each $n \geq z$ we can construct a binary experiment π''_{θ_n} with $E_{\langle \pi_{\theta_n}|\theta_n \rangle}[W(\beta)] > E_{\langle \hat{\pi}|\theta_n \rangle}[W(\beta)] \geq E_{\langle \pi^*_{\theta_N}|\theta_n \rangle}[W(\beta)]$ such that $\eta(\pi''_{\theta_n}) = \eta(\hat{\pi}) \geq \eta(\pi^*_{\theta_{z-1}})$ and (33) holds with equality. Then by Proposition 2, $D'' = \{\pi''_\theta\}_{\theta \geq \theta_z} \cup D^* \setminus \{\pi^*_\theta\}_{\theta \geq \theta_z}$ satisfies (EC) and ((M)). But since $E_{\langle \pi''_{\theta_n}|\theta_n \rangle}[W(\beta)] > E_{\langle \pi^*_{\theta_N}|\theta_n \rangle}[W(\beta)]$ for each $n \geq z$, D'' attains a higher value than D^* in (COPT), a contradiction.

Therefore, in all possible cases there is no experiment which Pareto dominates $\pi^*_{\theta_N}$ for type θ_N. □

Proof of Proposition 5 (Optimal Menu vs Complete Information Benchmark) This argument is analogous to Theorem 3(ii). Suppose that π_{θ_n} is binary, on the Pareto frontier for type θ_n, $E_{\langle \pi_{\theta_n}|\theta_n \rangle}[W(\beta)] \geq E_{\langle \pi^*_{\theta_n}|\theta^*_n \rangle}[W(\beta)]$, and $E_{\langle \pi_{\theta_n}|\theta_n \rangle}[U(\beta) - G(\beta|\theta_n)] \geq 0$. Since $E_{\langle \pi_{\theta_n}|\theta_n \rangle}[U(\beta) - G(\beta|\theta_n)] \geq 0$, we must have $C(\pi_{\theta_n}) < \infty$.

Suppose that $E_{\langle \pi_{\theta_n}|\theta_n \rangle}[W(\beta)] \geq E_{\langle \pi^*_{\theta_n}|\theta_n \rangle}[W(\beta)]$. Analogously to Lemma 18, we argue that $\pi^*_{\theta_n}$ cannot be Blackwell-more informative than π_{θ_n}. Suppose toward a contradiction that $\pi^*_{\theta_n}$ is Blackwell-more informative than π_{θ_n}: $\pi_{\theta_n}(\bar{s}|0) \geq \pi^*_{\theta_n}(\bar{s}|0)$, while $\pi_{\theta_n}(\underline{s}|1) \geq \pi^*_{\theta_n}(\underline{s}|1)$. Since π_{θ_n} and $\pi^*_{\theta_n}$ are distinct, one of these inequalities must be strict. Then $\langle \pi^*_{\theta_n}|\theta_n \rangle$ is a mean-preserving spread of $\langle \pi_{\theta_n}|\theta_n \rangle$. If $\overline{\beta} < b$, then (IRθ) fails, a contradiction by Theorem 2. If $\overline{\beta} = b$, then since $\overline{\beta}^* > b$, it is immediate that $E_{\langle \pi_{\theta_n}|\theta_n \rangle}[W(\beta)] = 0 < E_{\langle \pi^*_{\theta_n}|\theta_n \rangle}[W(\beta)]$, a contradiction. Finally, if $\overline{\beta} > b$, then since W is convex and strictly convex about the threshold, it follows from Lemma 13 that $E_{\langle \pi_{\theta_n}|\theta_n \rangle}[W(\beta)] < E_{\langle \pi^*_{\theta_n}|\theta_n \rangle}[W(\beta)]$, a contradiction.

Suppose toward a contradiction that $\pi_{\theta_n}(\underline{s}|1) \geq \pi^*_{\theta_n}(\underline{s}|1)$; then $\pi^*_{\theta_n} \not>_B \pi_{\theta_n}$. Since $\pi^*_{\theta_n} \not\geq_B \pi_{\theta_n}$, we must have $\pi_{\theta_n}(\underline{s}|0) > \pi^*_{\theta_n}(\underline{s}|0)$. Let $z = \max\{i \leq n \mid \delta_i = 0 \text{ or } i = 0\}$. That is, θ_z is the largest type less than or equal to θ_n such that the monotonicity constraint does not bind. Then by Lemma 5, $\pi^*_{\theta_n} = \pi^*_{\theta_z}$.

We show that there must be some $\pi' \in \Pi$ that is on the Pareto frontier for type θ_z for which $\pi'(\underline{s}|1) \geq \pi^*_{\theta_n}(\underline{s}|1)$, $\pi'(\underline{s}|0) > \pi^*_{\theta_n}(\underline{s}|0)$, and $C(\pi') < \infty$. If π_{θ_n} is on the Pareto frontier for type θ_z, this is immediate by setting $\pi' = \pi_{\theta_n}$; suppose not. Then there is some $\pi \in \Pi$ that is a Pareto improvement on π_{θ_n} for type θ_z; choose π' to be such a π that is on the Pareto frontier for type θ_z, and note that since $C(\pi_{\theta_n}) < \infty$, we must have $C(\pi') < \infty$ as well. If $\pi'(\underline{s}|1) < \pi_{\theta_n}(\underline{s}|1)$, then by Lemma 19, π' is also a Pareto improvement on π_{θ_n} for type θ_n, a contradiction since π_{θ_n} is on the Pareto frontier for type θ_n. So we must have $\pi'(\underline{s}|1) \geq \pi_{\theta_n}(\underline{s}|1) \geq \pi^*_{\theta_n}(\underline{s}|1)$. By Lemma 18, $\pi_{\theta_n} \not\geq_B \pi'$, so $\pi'(\underline{s}|0) > \pi_{\theta_n}(\underline{s}|0) > \pi^*_{\theta_n}(\underline{s}|0)$.

Let $\text{supp}\langle \pi^*_{\theta_n} | \theta_z \rangle = \{\underline{\beta}^*, \overline{\beta}^*\}$ with $\underline{\beta}^* < \theta_z < \overline{\beta}^*$, and $\text{supp}\langle \pi' | \theta_z \rangle = \{\underline{\beta}', \overline{\beta}'\}$ with $\underline{\beta} < \theta_z < \overline{\beta}$. Since $\pi'(\underline{s}|1) \geq \pi^*_{\theta_n}(\underline{s}|1)$ and $\pi'(\underline{s}|0) > \pi^*_{\theta_n}(\underline{s}|0)$, π' and $\pi^*_{\theta_n}$ cannot be Blackwell-ranked. Then one of two cases must hold: (a) $\underline{\beta} \leq \underline{\beta}^*$ and $\overline{\beta} \leq \overline{\beta}^*$, with one inequality strict; or (b) $\underline{\beta} \geq \underline{\beta}^*$ and $\overline{\beta} \geq \overline{\beta}^*$, with one inequality strict.

Suppose that (b) holds. By Lemma 15, since $\theta_z \in \tilde{\Theta}$, we must have $\overline{\beta}^* > b$. By Proposition 3, since π' is on the Pareto frontier for type θ_z, there exist Pareto weights $\lambda_p, \lambda_a \geq 0$ such that $\langle \pi' | \theta_z \rangle$ solves (SPPθ) for $\theta = \theta_z$. Since $C(\pi') < \infty$, π' is not fully informative: $\pi' \not\sim_B \pi_\infty$. Then by Corollary 1, either equation (2) or (3) from Section 4.2 holds for $\theta = \theta_z$. Moreover, since $\pi^*_{\theta_n}$ solves (TBTθ_n) for $\theta = \theta_z$, (6) holds (with θ_n replaced by θ_z) at $(\underline{\beta}^*, \overline{\beta}^*)$. Then we have

$$
\begin{aligned}
R(b, \theta_z)/\lambda^*_z &= u - G(\overline{\beta}^* | \theta_z) - G'(\overline{\beta}^* | \theta_z) + G(\underline{\beta}^* | \theta_z) + G'(\underline{\beta}^* | \theta_z)(b - \underline{\beta}^*) \\
&< u - G(\overline{\beta} | \theta_z) - G'(\overline{\beta} | \theta_z) + G(\underline{\beta} | \theta_z) + G'(\underline{\beta} | \theta_z)(b - \underline{\beta}) = 0, \qquad (34)
\end{aligned}
$$

but this contradicts Lemma 17. So it must be the case that (a) $\underline{\beta} \leq \underline{\beta}^*$ and $\overline{\beta} \leq \overline{\beta}^*$, with one inequality strict. Then since Bayesian updating is multiplicative in likelihood ratios, and $\overline{\beta} \leq \overline{\beta}^*$, we must have $\pi^*_{\theta_n}(\bar{s}|1)/\pi^*_{\theta_n}(\bar{s}|0) \geq \pi'(\bar{s}|1)/\pi'(\bar{s}|0)$. Since $\pi'(\underline{s}|0) > \pi^*_{\theta_n}(\underline{s}|0)$, we have $\pi'(\bar{s}|0) < \pi^*_{\theta_n}(\bar{s}|0)$, and hence $\pi^*_{\theta_n}(\bar{s}|1) > \pi'(\bar{s}|1)$, a contradiction. □