

Delegating Experiments

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Online Appendix

Lemma OA.1. *Let $v : [0,1] \rightarrow \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function; let $\alpha \in \text{ri}(\text{dom } v)$; and let V be the concavification of v . If $\tau^* \in \arg \max_{\tau} \{E_{\tau} v(\beta) \text{ s.t. } E_{\tau} \beta = \alpha\}$, then for any $\beta \in \text{supp } \tau^*$,*

i. If v is differentiable at β , then $v'(\beta) = V'(\alpha)$.

ii. If v is strictly concave on $[0, z)$ and $(1 - z, 1]$ and continuously differentiable on $(0, z)$ and $(1 - z, 1)$ for some $z > 0$, $\lim_{x \rightarrow 0} v'(x) = \infty$, and $\lim_{x \rightarrow 1} v'(x) = -\infty$, then $\beta \notin \{0, 1\}$.

Proof. By Proposition 3 in Yoder (2023), $v(\beta) = V(\beta)$, and V is affine on $\text{conv}(\text{supp } \pi^*)$. Then $v(\beta) = V(\alpha) + V'(\alpha)(\beta - \alpha)$. Since V is concave, for all $\beta' \in [0, 1]$ we have

$$\begin{aligned} v(\beta') &\leq V(\alpha) + V'(\alpha)(\beta' - \alpha), \\ &= v(\beta) + V'(\alpha)(\beta' - \beta). \end{aligned} \tag{OA.1}$$

Then for $0 < \epsilon < \min\{\beta, 1 - \beta\}$,

$$\begin{aligned} v(\beta + \epsilon) &\leq v(\beta) + V'(\alpha)\epsilon, \\ v(\beta - \epsilon) &\leq v(\beta) - V'(\alpha)\epsilon, \\ \Rightarrow \frac{v(\beta + \epsilon) - v(\beta)}{\epsilon} &\leq V'(\alpha) \leq \frac{v(\beta) - v(\beta - \epsilon)}{\epsilon}. \end{aligned}$$

Then if v is differentiable at β , (i) follows by the squeeze theorem.

For (ii), suppose $\beta = 0$, v is continuously differentiable on $(0, z)$ and strictly concave on $[0, z)$ for some $z > 0$, and $\lim_{x \rightarrow 0} v'(x) = \infty$. Since $\lim_{x \rightarrow 0} v'(x) = \infty$, there exists $y \in (0, \min\{z, \alpha\})$ such that $v'(y) > V'(\alpha)$. Since v is strictly concave on $[0, z)$, $v(0) < v(y) - v'(y)y$. By Proposition 3 in Yoder (2023), $v(0) = V(0)$. And by definition, $V(y) \geq v(y)$. Then by (OA.1), we have

$$V(0) + V'(\alpha)y = v(0) + V'(\alpha)y < v(0) + v'(y)y < v(0) + (v(y) - v(0)) = v(y) \leq V(y),$$

a contradiction since V is concave. A similar argument shows that $\beta \neq 1$. \square

Lemma OA.2 (A Necessary and Sufficient Tangent Line Condition). *Suppose that $v : [0, 1] \rightarrow \mathbb{R} \cup \{-\infty\}$ is continuous and strictly concave on $[0, b)$ and on $[b, 1]$, continuously differentiable on $(0, b)$ and $(b, 1)$, that $\lim_{\beta \rightarrow 0} v'(\beta) = \infty$, and $\lim_{\beta \rightarrow 1} v'(\beta) = -\infty$, and that $\alpha \in (0, b)$. Then $\max_{\tau} \{E_{\tau} v(\beta) \text{ s.t. } E_{\tau} \beta = \alpha\}$ has a solution τ^* , it is unique, and*

i. *If there exist $\underline{\beta}, \bar{\beta}$ with $b, \alpha \in (\underline{\beta}, \bar{\beta})$ and $\frac{v(\bar{\beta}) - v(\underline{\beta})}{\bar{\beta} - \underline{\beta}} = v'(\underline{\beta}) = v'(\bar{\beta})$, then $\text{supp } \tau^* = \{\underline{\beta}, \bar{\beta}\}$;*

ii. *If not, and there exists $\underline{\beta} < \alpha < b$ with $\frac{v(b) - v(\underline{\beta})}{b - \underline{\beta}} = v'(\underline{\beta})$, then $\text{supp } \tau^* = \{\underline{\beta}, b\}$;*

iii. *If the conditions in both (i) and (ii) fail, then $\text{supp } \tau^* = \{\alpha\}$.*

Proof. We begin by showing that Lemma OA.2 holds under the additional assumption that v is upper semicontinuous (or equivalently, $\lim_{\beta \uparrow b} v(\beta) \leq v(b)$). Since solutions to persuasion problems are preserved under upper semicontinuous hulls (Lemma S.5 in Yoder (2022)), the uniqueness of the solution to the upper semicontinuous hull problem guarantees that it is also the unique solution to the (possibly not upper semicontinuous) original problem. We then show existence by showing that if v is not upper semicontinuous, then the solution to the u.s.c. hull problem must be case (i) or (iii), in which case it is also a solution to the original problem.

To prove Lemma OA.2 under upper semicontinuity, we first show existence, and then show that the conditions in (i),(ii), and (iii) imply that τ^* has support $\{\underline{\beta}, \bar{\beta}\}$, $\{\underline{\beta}, b\}$, and $\{\alpha\}$, respectively. Then, we show that in each case, τ^* is the unique solution to $\max_{\tau} \{E_{\tau} v(\beta) \text{ s.t. } E_{\tau} \beta = \alpha\}$.

Existence of τ^* under upper semicontinuity. Follows from, e.g., Proposition 3 in Yoder (2023).

Support of τ^* . (i): Suppose that condition (i) holds, and v is upper semicontinuous. Then by strict concavity, $v(\underline{\beta}) + \frac{v(\bar{\beta}) - v(\underline{\beta})}{\bar{\beta} - \underline{\beta}}(\beta - \underline{\beta}) = v(\underline{\beta}) + v'(\underline{\beta})(\beta - \underline{\beta}) \geq v(\beta)$ for all $\beta \in [0, b)$, and $v(\underline{\beta}) + \frac{v(\bar{\beta}) - v(\underline{\beta})}{\bar{\beta} - \underline{\beta}}(\beta - \underline{\beta}) = v(\bar{\beta}) + v'(\bar{\beta})(\beta - \bar{\beta}) \geq v(\beta)$ for all $\beta \in [b, 1]$. Then by Proposition S.7 in Yoder (2022), the unique τ^* with $\text{supp } \tau^* = \{\underline{\beta}, \bar{\beta}\}$ solves $\max_{\tau} \{E_{\tau} v(\beta) \text{ s.t. } E_{\tau} \beta = \alpha\}$.

(ii),(iii): Suppose that condition (i) fails, and v is upper semicontinuous. By Proposition 3 in Yoder (2023), $\max_{\tau} \{E_{\tau} v(\beta) \text{ s.t. } E_{\tau} \beta = \alpha\}$ has a solution τ^* with $|\text{supp } \tau^*| \leq 2$. Then by Proposition S.7 in Yoder (2022), either

A. $\text{supp } \tau^* = \{\alpha\}$, or

B. $\text{supp } \tau^* = \{\underline{\beta}', \bar{\beta}'\}$ and $v(\beta) \leq v(\underline{\beta}') + \frac{v(\bar{\beta}') - v(\underline{\beta}')}{\bar{\beta}' - \underline{\beta}'}(\beta - \underline{\beta}')$ for all $\beta \in [0, 1]$.

Suppose that (B) holds. By Lemma 2 in Yoder (2022), we must have $\underline{\beta}' < \alpha < \bar{\beta}'$. Since v is strictly concave on $[0, b)$ and $[b, 1]$, we cannot have $\{\underline{\beta}', \bar{\beta}'\} \subset [0, b)$ or $\{\underline{\beta}', \bar{\beta}'\} \subset [b, 1]$. Then $\underline{\beta}' < \alpha, b \leq \bar{\beta}'$. Moreover, by Lemma OA.1, since v is continuously differentiable on $(0, b)$ and $(b, 1)$, and $\lim_{\beta \rightarrow 0} v'(\beta) = \infty$ and $\lim_{\beta \rightarrow 1} v'(\beta) = -\infty$, we have $0 < \underline{\beta}' < \alpha, b \leq \bar{\beta}' < 1$, and $\frac{v(\bar{\beta}') - v(\underline{\beta}')}{\bar{\beta}' - \underline{\beta}'} = v'(\beta)$ for each $\beta \in \{\underline{\beta}', \bar{\beta}'\}$ with $\beta \neq b$. Then since (i) fails, we must have $\bar{\beta}' = b$.

Then if condition (i) fails and (B) holds, condition (ii) follows. By contrapositive, if conditions (i) and (ii) both fail, we must have $\text{supp } \tau^* = \{\alpha\}$; (iii) follows.

Now suppose that condition (i) fails and condition (ii) holds. Let τ' be such that $\text{supp } \tau' = \{\underline{\beta}, b\}$ and $E_{\tau'}\beta = \alpha$. Since v is strictly concave on $[0, b)$, $E_{\tau'}v(\beta) = E_{\tau'}v(\beta) = v(\underline{\beta}) + \frac{v(b) - v(\underline{\beta})}{b - \underline{\beta}}(\alpha - \underline{\beta}) = v(\underline{\beta}) + v'(\underline{\beta})(\alpha - \underline{\beta}) > v(\alpha)$. Then $\text{supp } \tau^* \neq \{\alpha\}$, and so (B) must hold. Then we must have $\text{supp } \tau^* = \{\underline{\beta}', b\}$ for some $\underline{\beta}' < \alpha$. Suppose toward a contradiction that $\underline{\beta}' \neq \underline{\beta}$. Then since v is strictly concave on $[0, b)$, $v(\underline{\beta}') < v(\underline{\beta}) + v'(\underline{\beta})(\underline{\beta}' - \underline{\beta}) = v(\underline{\beta}) + \frac{v(b) - v(\underline{\beta})}{b - \underline{\beta}}(\underline{\beta}' - \underline{\beta})$. Then $E_{\tau^*}[v(\beta)] = E_{\tau'}[v(\beta)] < E_{\tau^*} \left[v(\underline{\beta}) + \frac{v(b) - v(\underline{\beta})}{b - \underline{\beta}}(\beta - \underline{\beta}) \right] = v(\underline{\beta}) + \frac{v(b) - v(\underline{\beta})}{b - \underline{\beta}}(\alpha - \underline{\beta}) = E_{\tau'}[v(\beta)] = E_{\tau^*}[v(\beta)]$, a contradiction since $\tau^* \in \arg \max_{\tau} \{E_{\tau}v(\beta) \text{ s.t. } E_{\tau}\beta = \alpha\}$. It follows that $\text{supp } \tau^* = \{\underline{\beta}, b\}$.

Uniqueness of τ^* . First suppose that condition (i) holds, and $\text{supp } \tau^* = \{\underline{\beta}, \bar{\beta}\}$ for $0 < \underline{\beta} < \alpha, b < \bar{\beta} < 1$. By strict concavity, $v(\underline{\beta}) + \frac{v(\bar{\beta}) - v(\underline{\beta})}{\bar{\beta} - \underline{\beta}}(\beta - \underline{\beta}) = v(\underline{\beta}) + v'(\underline{\beta})(\beta - \underline{\beta}) > v(\beta)$ for all $\beta \in [0, b) \setminus \{\underline{\beta}\}$, and $v(\underline{\beta}) + \frac{v(\bar{\beta}) - v(\underline{\beta})}{\bar{\beta} - \underline{\beta}}(\beta - \underline{\beta}) = v(\bar{\beta}) + v'(\bar{\beta})(\beta - \bar{\beta}) > v(\beta)$ for all $\beta \in [b, 1] \setminus \{\bar{\beta}\}$.

Then τ^* is the unique solution to $\max_{\tau} \{E_{\tau}v(\beta) \text{ s.t. } E_{\tau}\beta = \alpha\}$: Suppose that $\text{supp } \tau \neq \{\underline{\beta}, \bar{\beta}\}$. If $|\text{supp } \tau|$ is finite, then since $v(\beta) < v(\underline{\beta}) + \frac{v(\bar{\beta}) - v(\underline{\beta})}{\bar{\beta} - \underline{\beta}}(\beta - \underline{\beta})$ for all $\beta \notin \{\underline{\beta}, \bar{\beta}\}$, we have $E_{\tau}[v(\beta)] < E_{\tau^*} \left[v(\underline{\beta}) + \frac{v(\bar{\beta}) - v(\underline{\beta})}{\bar{\beta} - \underline{\beta}}(\beta - \underline{\beta}) \right] = E_{\tau^*}[v(\beta)]$. Otherwise, either τ is nondegenerate on $[0, b)$ or it is nondegenerate on $[b, 1]$. Suppose the former; then since v

is strictly concave on $[0, b)$ and $[b, 1]$, by Jensen's inequality, we have

$$\begin{aligned}
& E_\tau[v(\beta)|\beta < b] < v(E_\tau[\beta|\beta < b]) \\
& E_\tau[v(\beta)|\beta \geq b] \leq v(E_\tau[\beta|\beta \geq b]) \\
\Rightarrow E_\tau[v(\beta)] &= \frac{E_\tau[v(\beta)|\beta < b]\tau([0, b))}{+E_\tau[v(\beta)|\beta \geq b]\tau([b, 1])} < \frac{v(E_\tau[\beta|\beta < b])\tau([0, b))}{+v(E_\tau[\beta|\beta \geq b])\tau([b, 1])} \\
& \leq \left(v(\underline{\beta}) + \frac{v(\bar{\beta})-v(\underline{\beta})}{\bar{\beta}-\underline{\beta}}(E_\tau[\beta|\beta < b] - \underline{\beta}) \right) \tau([0, b)) \\
& \quad + \left(v(\underline{\beta}) + \frac{v(\bar{\beta})-v(\underline{\beta})}{\bar{\beta}-\underline{\beta}}(E_\tau[\beta|\beta \geq b] - \underline{\beta}) \right) \tau([b, 1]) \\
& = v(\underline{\beta}) + \frac{v(\bar{\beta}) - v(\underline{\beta})}{\bar{\beta} - \underline{\beta}}(\alpha - \underline{\beta}) = E_{\tau^*}[v(\beta)].
\end{aligned}$$

A symmetric argument shows that $E_\tau[v(\beta)] < E_{\tau^*}[v(\beta)]$ when τ is non-degenerate on $[b, 1]$.

For case (ii), the same argument with $\bar{\beta} = b$ shows that τ^* is unique.

For case (iii), where $\text{supp } \tau^* = \{\alpha\}$: First note that by Proposition 3 in Yoder (2023), $V(\alpha) = v(\alpha)$, and by Lemma OA.1, if $\alpha \neq b$, $v'(\alpha) = V'(\alpha)$. Suppose that $\tau' \in \arg \max_\tau \{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$ and $\text{supp } \tau' \neq \{\alpha\}$. Then since $E_{\tau'} \beta = \alpha$, there must be some $\beta, \beta' \in \text{supp } \tau'$ with $\beta < \alpha < \beta'$. If $\alpha < b$, then by Lemma OA.1, since v is differentiable and strictly concave on $(0, b)$, $v'(\beta) = V'(\alpha) = v'(\alpha) < v'(\beta)$, a contradiction. Alternatively, if $\alpha \geq b$, then by Lemma OA.1, since v is differentiable on $(b, 1)$, $v'(\beta') = V'(\alpha)$. By Proposition 3 in Yoder (2023), V is affine on $\text{conv}(\text{supp } \tau') \supseteq [\alpha, \beta')$, so $v(\alpha) = V(\alpha) = V(\beta') + V'(\beta')(\alpha - \beta') = v(\beta') + v'(\beta')(\alpha - \beta')$, a contradiction since v is strictly concave on $(b, 1)$.

Extension to non-upper semicontinuous v . Suppose that v is not upper semicontinuous. Then $\lim_{\beta \uparrow b} v(\beta) > v(b)$. Let \bar{v} be the upper semicontinuous hull of v ; we have $\bar{v}(b) = \lim_{\beta \uparrow b} v(\beta)$ and $\bar{v}(\beta) = v(\beta)$ for all $\beta \neq b$, so that \bar{v} is strictly concave and continuous on $[0, b]$ and $(b, 1]$. Since Lemma OA.2 holds under upper semicontinuity, there exists a unique $\tau^* \in \arg \max_\tau \{E_\tau \bar{v}(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$, and

- i. If there exist $\underline{\beta}, \bar{\beta}$ with $b, \alpha \in (\underline{\beta}, \bar{\beta})$ and $\frac{v(\bar{\beta})-v(\underline{\beta})}{\bar{\beta}-\underline{\beta}} = v'(\underline{\beta}) = v'(\bar{\beta})$, then $\text{supp } \tau^* = \{\underline{\beta}, \bar{\beta}\}$;
- ii.' If not, and there exists $\underline{\beta} < \alpha$ with $\frac{\bar{v}(b)-v(\underline{\beta})}{b-\underline{\beta}} = v'(\underline{\beta})$, then $\text{supp } \tau^* = \{\underline{\beta}, b\}$;
- iii.' If the conditions in both (i) and (ii') fail, then $\text{supp } \tau^* = \{\alpha\}$.

But (ii') cannot hold: Suppose it does. Then $\frac{\bar{v}(b)-\bar{v}(\underline{\beta})}{b-\underline{\beta}} = \frac{\bar{v}(b)-v(\underline{\beta})}{b-\underline{\beta}} = \bar{v}'(\underline{\beta})$, a contradiction since \bar{v} is strictly concave and continuous on $[0, b]$.

Thus, either (i) holds, or $\text{supp } \tau^* = \{\alpha\}$; in either case, $b \notin \text{supp } \tau^*$. Then by definition of \bar{v} , for all τ with $E_\tau \beta = \alpha$, we have

$$E_{\tau^*}[v(\beta)] = E_{\tau^*}[\bar{v}(\beta)] \geq E_\tau[\bar{v}(\beta)] \geq E_\tau[v(\beta)],$$

and so $\tau^* \in \arg \max_\tau \{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$. Hence $\arg \max_\tau \{E_\tau \bar{v}(\beta) \text{ s.t. } E_\tau \beta = \alpha\} \subseteq \arg \max_\tau \{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$; by Lemma S.5 in Yoder (2022), the two sets are equal, and τ^* uniquely solves $\max_\tau \{E_\tau v(\beta) \text{ s.t. } E_\tau \beta = \alpha\}$.

It remains to be shown that (ii) cannot hold, and so τ^* is pinned down by (i),(ii), and (iii), rather than by (i),(ii'), and (iii'). Suppose that it does, and there exists $\underline{\beta} < \alpha < b$ with $v'(\underline{\beta}) = \frac{v(b)-v(\underline{\beta})}{b-\underline{\beta}}$. Since v is strictly concave on $[0, b)$, $v(\underline{\beta}) + v'(\underline{\beta})(b - \underline{\beta}) > \lim_{\beta \uparrow b} v(\beta) > v(b)$. Then $v'(\underline{\beta}) > \frac{v(b)-v(\underline{\beta})}{b-\underline{\beta}}$, a contradiction. \square

Lemma OA.3. *Suppose that $v : [0, 1] \rightarrow \mathbb{R} \cup \{-\infty\}$ is concave, and is strictly concave about the threshold: For any $\beta < b < \beta'$ and any $x \in \partial v(\beta)$, $y \in \partial v(\beta')$, we have $x > y$. If $\text{supp } \tau = \{\underline{\beta}, \bar{\beta}\}$, $\text{supp } \tau' = \{\underline{\beta}', \bar{\beta}'\}$, $E_\tau \beta = E_{\tau'} \beta$, $\underline{\beta} < b < \bar{\beta}$, and τ' is a mean-preserving spread of τ , then $E_{\tau'}[v(\beta)] < E_\tau[v(\beta)]$.*

Proof. Let $v_1(\beta) = v(\underline{\beta}) + \frac{v(\bar{\beta})-v(\underline{\beta})}{\bar{\beta}-\underline{\beta}}(\beta - \underline{\beta})$ be the secant line to v at $\underline{\beta}$ and $\bar{\beta}$. Then we have $E_{\tau'}[v_1(\beta)] = E_\tau[v_1(\beta)] = E_\tau[v(\beta)]$.

Since v is concave, for any $x \in \partial v(\underline{\beta})$ and $y \in \partial v(\bar{\beta})$, we have $x \geq \frac{v(\bar{\beta})-v(\underline{\beta})}{\bar{\beta}-\underline{\beta}} \geq y$. If either holds with equality, then since v is concave, it must be affine on $[\underline{\beta}, \bar{\beta}]$ — a contradiction since $\underline{\beta} < b < \bar{\beta}$ and v is strictly concave about the threshold. So we must have $x > \frac{v(\bar{\beta})-v(\underline{\beta})}{\bar{\beta}-\underline{\beta}} > y$ for all $x \in \partial v(\underline{\beta})$ and $y \in \partial v(\bar{\beta})$. Choose such an x and y ; since v is concave,

$$\begin{aligned} v(\bar{\beta}') &\leq v(\bar{\beta}) + y(\bar{\beta}' - \bar{\beta}) < v_1(\bar{\beta}'), & \text{if } \bar{\beta} \neq \bar{\beta}'; \\ v(\underline{\beta}') &\leq v(\underline{\beta}) + x(\underline{\beta}' - \underline{\beta}) < v_1(\underline{\beta}'), & \text{if } \underline{\beta} \neq \underline{\beta}'. \end{aligned}$$

Since τ and τ' are distinct, it follows that $E_{\tau'}[v(\beta)] < E_{\tau'}[v_1(\beta)] = E_\tau[v(\beta)]$, as desired. \square

Implementability

Lemma OA.4 (Deterministic Without Loss). *The optimal mechanism that solves (OPT) is deterministic without loss of generality. That is, $\chi^* : \Theta \rightarrow \Pi \times \mathcal{A}$.*

Proof. The principal's program (OPT) is linear in probabilities of allocations in the support

of $\chi(\theta)$, since

$$\int_{\Pi \times \mathcal{A}} C(\pi) dx(\theta) = C \left(\int_{\Pi \times \mathcal{A}} \pi dx(\theta) \right)$$

by Theorem 1 of Pomatto et al. (2023). Therefore, since the principal's program is linear, it is without loss of optimality to restrict attention to deterministic mechanisms. \square

Proof of Lemma 1 (Agent's Distribution of Principal's Posteriors) Let $B \subseteq [0, 1]$. If $\langle \pi | \alpha \rangle(B) = 0$, the statement follows trivially. Otherwise, define $S_B \subseteq S$ to be the set of signal realizations from π that induce posterior beliefs $\beta \in B$ to a principal with prior α . Then the probability with which an agent of type θ believes that S_B will occur is

$$\theta \pi(S_B | \omega = 1) + (1 - \theta) \pi(S_B | \omega = 0) = \frac{\theta}{\alpha} \cdot \alpha \pi(S_B | \omega = 1) + \frac{1 - \theta}{1 - \alpha} \cdot (1 - \alpha) \pi(S_B | \omega = 0).$$

By Bayes' rule, note that

$$\int_B \beta d\langle \pi | \alpha \rangle(\beta) = \pi(S_B | \omega = 1) \alpha; \quad 1 - \int_B \beta d\langle \pi | \alpha \rangle(\beta) = \pi(S_B | \omega = 0) (1 - \alpha).$$

So, the probability with which an agent of type θ believes that S_B will occur simplifies to

$$\int_B \left(\frac{\theta}{\alpha} \beta + \frac{1 - \theta}{1 - \alpha} (1 - \beta) \right) d\langle \pi | \alpha \rangle(\beta)$$

as desired. \square

The Principal's Problem

Proof of Lemma 3 (Exclusion Is At The Low End) Suppose for sake of contradiction that there exist $\theta, \theta' \in \Theta$ with $\theta' > \theta$ such that $\pi_{\theta'}^* \sim_B \pi_0$ but $\pi_{\theta}^* \not\sim_B \pi_0$. Since D^* solves (COPT), it satisfies (M) and (EC); since it satisfies (EC), it is individually rational. Then since D^* is binary by Theorem 2, Proposition 2 implies that $\eta(\pi_{\theta'}^*) \geq \eta(\pi_{\theta}^*)$. Since $\pi_{\theta'}^* \sim \pi_0$, $\eta(\pi_{\theta'}^*) = 0$, and hence $\eta(\pi_{\theta}^*) = 0$. Then, $\pi_{\theta}^*(\cdot | 1) = \pi_{\theta}^*(\cdot | 0)$, and so $\pi_{\theta}^* \sim \pi_0$, a contradiction. \square

Proof of Lemma 4 (Experiments Where Monotonicity Binds) Let $d^* = \{(x_n^*, y_n^*)\}_{n=0}^N$ be the (unique, by Lemma 8) solution to (OPT2); by Claims T2.2(ii) and T2.3, $\pi_{\theta_n}^*(\bar{s} | 1) = x_n^*$ and $\pi_{\theta_n}^*(\bar{s} | 0) = y_n^*$ for each $n \in \{0, \dots, N\}$. It thus suffices to show that if $j < k$ and $x_k^* - y_k^* = x_j^* - y_j^*$, then $(x_i^*, y_i^*) = (x_j^*, y_j^*)$ for each $i \in \{j, \dots, k\}$.

Suppose $j < k$ and $x_k^* - y_k^* = x_j^* - y_j^*$. Since d^* satisfies the monotonicity constraints

in (OPT2), for each $i \in \{j, \dots, k\}$,

$$x_i^* - y_i^* = x_k^* - y_k^*.$$

Fix $i \in \{j, \dots, k\}$. Since d^* is feasible in (OPT2),

$$\begin{aligned} U_b(x_k^*, y_k^*, \theta_k) &= U_b(x_i^*, y_i^*, \theta_i) + u \sum_{r=i}^{k-1} (\theta_{r+1} - \theta_r)(x_r^* - y_r^*) \\ &= U_b(x_i^*, y_i^*, \theta_i) + u(\theta_k - \theta_i)(x_i^* - y_i^*) \\ &= U_b(x_i^*, y_i^*, \theta_k). \end{aligned} \tag{OA.2}$$

Suppose toward a contradiction that $(x_i^*, y_i^*) \neq (x_k^*, y_k^*)$. Since $x_i^* - y_i^* = x_k^* - y_k^*$, it must be that $y_i^* \neq y_k^*$.

If $y_i^* < y_k^*$, let $\hat{d} = \{(x_\ell^*, y_\ell^*)\}_{\ell \neq k} \cup \{(x_i^*, y_i^*)\}$ be obtained from d^* by replacing (x_k^*, y_k^*) with (x_i^*, y_i^*) and leaving every other pair unchanged. By (OA.2), and since $x_i^* - y_i^* = x_k^* - y_k^*$ and d^* is feasible in (OPT2), \hat{d} is feasible in (OPT2). Moreover,

$$W_b(x_i^*, y_i^*, \theta_k) - W_b(x_k^*, y_k^*, \theta_k) = (y_i^* - y_k^*)(\theta_k w_1 + (1 - \theta_k)w_0) > 0,$$

since $\theta_k < b$ and $y_i^* < y_k^*$. This contradicts the optimality of d^* in (OPT2).

If $y_i^* > y_k^*$, let $\hat{d} = \{(x_\ell^*, y_\ell^*)\}_{\ell \neq i} \cup \{(x_k^*, y_k^*)\}$ be obtained from d^* by replacing (x_i^*, y_i^*) with (x_k^*, y_k^*) ; a symmetric argument shows that \hat{d} is feasible in (OPT2) and

$$W_b(x_k^*, y_k^*, \theta_i) - W_b(x_i^*, y_i^*, \theta_i) = (y_k^* - y_i^*)(\theta_i w_1 + (1 - \theta_i)w_0) > 0,$$

again contradicting the optimality of d^* in (OPT2). Therefore $y_i^* = y_k^*$, and then $x_i^* = x_k^*$ as well, as desired. \square

Lemma OA.5. *The “distortion term” $R(b, \theta_n)/\lambda_n^*$ is strictly positive whenever $\underline{\theta} \leq \theta_n < \theta_N$ and $\delta_n^* = 0$.*

Proof. Suppose that $\underline{\theta} \leq \theta_n < \theta_N$ and $\delta_n^* = 0$. By Theorem 2, $\lambda_i^* = 0$ implies $\pi_{\theta_i}^* \sim_B \pi_0$; hence $\lambda_i^* > 0$ for all $\theta_i \geq \underline{\theta}$. Then we have

$$\begin{aligned} R(b, \theta_n)/\lambda_n^* &= \left(\frac{b - \theta_n}{\theta_n(1 - \theta_n)} \right) u \left(\left(\delta_{n+1}^* + (\theta_{n+1} - \theta_n) \sum_{i=n+1}^N \lambda_i^* \right) \mathbf{1}_{n < N} - \delta_n^* \right) / \lambda_n^* \\ &= \left(\frac{b - \theta_n}{\theta_n(1 - \theta_n)} \right) u \left(\delta_{n+1}^* + (\theta_{n+1} - \theta_n) \sum_{i=n+1}^N \lambda_i^* \right) / \lambda_n^* > 0, \end{aligned}$$

since $\theta_n < b$ and $\delta_{n+1}^* \geq 0$. \square

The Pareto Frontier and Distortion

Lemma OA.6 (Monotonicity of Pareto Dominance). *If binary experiment π' Pareto dominates binary experiment π for type θ , $\pi'(\underline{s}|1) < \pi(\underline{s}|1)$, $W_b(\pi(\bar{s}|1), \pi(\bar{s}|0), \theta) \geq 0$, and $W_b(\pi'(\bar{s}|1), \pi'(\bar{s}|0), \theta) \geq 0$, then π' Pareto dominates π for every type $\theta' > \theta$.*

Proof. Let $x = \pi(\bar{s}|1)$, $y = \pi(\bar{s}|0)$, $x' = \pi'(\bar{s}|1)$, and $y' = \pi'(\bar{s}|0)$. Since $\pi'(\underline{s}|1) < \pi(\underline{s}|1)$, $x' > x$.

For any type $\hat{\theta} \geq \theta$, $W_b(x, y, \hat{\theta}) \geq 0$ and $W_b(x', y', \hat{\theta}) \geq 0$, since $\frac{\partial}{\partial t} W_b(x, y, t) = w_1 x - w_0 y \geq 0$ and $\frac{\partial}{\partial t} W_b(x', y', t) = w_1 x' - w_0 y' \geq 0$. Then by definition of b and W_b , if the principal's interim belief is $\hat{\theta}$, observing \bar{s} from π or π' must induce a posterior in $[b, 1]$ (if the experiment produces \bar{s} with positive probability). Then since $\hat{\theta} < b$, if the principal's interim belief is $\hat{\theta}$, observing \underline{s} from π or π' must induce a posterior in $[0, b)$. Then we have

$$\begin{aligned} \Delta_P(\hat{\theta}) &:= E_{\langle \pi' | \hat{\theta} \rangle} [W(\beta)] - E_{\langle \pi | \hat{\theta} \rangle} [W(\beta)] &&= W_b(x', y', \hat{\theta}) - W_b(x, y, \hat{\theta}), \\ \Delta_A(\hat{\theta}) &:= E_{\langle \pi' | \hat{\theta} \rangle} [U(\beta) - G(\beta | \hat{\theta})] - E_{\langle \pi | \hat{\theta} \rangle} [U(\beta) - G(\beta | \hat{\theta})] &&= U_b(x', y', \hat{\theta}) - U_b(x, y, \hat{\theta}), \end{aligned}$$

and π' Pareto dominates binary experiment π for type $\hat{\theta}$ iff $\Delta_P(\hat{\theta}) \geq 0$ and $\Delta_A(\hat{\theta}) \geq 0$, with one inequality strict.

We first show that $x' - y' \geq x - y$. If not, then $y' - y > x' - x > 0$, and

$$0 \leq \Delta_P(\theta) = \theta w_1 (x' - x) + (1 - \theta) w_0 (y' - y) < (x' - x)(\theta w_1 + (1 - \theta) w_0) < 0,$$

a contradiction since $\theta < b$. Hence, $x' - y' \geq x - y$.

Now fix a type $\theta' > \theta$. Since $x' - y' \geq x - y$,

$$\Delta_A(\theta') = \Delta_A(\theta) + (\theta' - \theta) u((x' - y') - (x - y)) \geq 0.$$

Moreover, $\Delta_P(\theta') > 0$: If $y' \leq y$, this follows from $x' > x$, $w_1 > 0$, and $w_0 < 0$; if $y' > y$, then

$$\Delta_P(\theta') = \Delta_P(\theta) + (\theta' - \theta)(w_1(x' - x) - w_0(y' - y)) > 0.$$

It follows that π' Pareto dominates π for type θ' . □

Proof of Proposition 5 (No Distortion at the Top) Let $d^* = \{(x_i^*, y_i^*)\}_{i=0}^N$ be the (unique, by Lemma 8) solution to (OPT2); let $\{\lambda_i^*\}_{i=0}^N$ and $\{\delta_i^*\}_{i=0}^N$ be the nonnegative Lagrange multipliers pinned down in Lemma 8. By Claims T2.2(ii) and T2.3, $\pi_{\theta_i^*}^*(\bar{s}|1) = x_i^*$ and $\pi_{\theta_i^*}^*(\bar{s}|0) = y_i^*$ for each $i \in \{0, \dots, N\}$. By Claim T2.2(i), $W_b(x_N^*, y_N^*, \theta_N) \geq 0$.

Suppose toward a contradiction that there is an experiment π that Pareto improves on $\pi_{\theta_N}^*$ for type θ_N ; without loss, we can take π to be on the Pareto frontier and thus

(by Corollary 2) binary. By Lemma 5 with $\alpha = \theta_N$, $h_1 = \theta_N$, $h_0 = 1 - \theta_N$, and $y = U_b(x_N^*, y_N^*, \theta_N)$, there exists a binary experiment π' such that $\pi'(\bar{s}|1) - \pi'(\bar{s}|0) = \pi(\bar{s}|1) - \pi(\bar{s}|0)$, $U_b(\pi'(\bar{s}|1), \pi'(\bar{s}|0), \theta_N) = U_b(x_N^*, y_N^*, \theta_N)$, and

$$W_b(\pi'(\bar{s}|1), \pi'(\bar{s}|0), \theta_N) \geq W_b(\pi(\bar{s}|1), \pi(\bar{s}|0), \theta_N) \geq W_b(x_N^*, y_N^*, \theta_N),$$

with one inequality strict: since π Pareto improves on $\pi_{\theta_N}^*$ for type θ_N , either the second inequality is strict, or $U_b(\pi(\bar{s}|1), \pi(\bar{s}|0), \theta_N) > U_b(x_N^*, y_N^*, \theta_N)$ and the first inequality is strict by Lemma 5. Write $x' = \pi'(\bar{s}|1)$ and $y' = \pi'(\bar{s}|0)$. Since $W_b(x', y', \theta_N) > W_b(x_N^*, y_N^*, \theta_N) \geq 0$, we must have $x' > y'$. We consider two cases.

Case I: $x' - y' \geq x_N^* - y_N^*$. Let $\hat{d} = \{(x_n^*, y_n^*)\}_{n=0}^{N-1} \cup \{(x', y')\}$. Since $U_b(x', y', \theta_N) = U_b(x_N^*, y_N^*, \theta_N)$ and $x' - y' \geq x_N^* - y_N^*$, \hat{d} is feasible in (OPT2) and achieves a strictly higher value of the objective, a contradiction.

Case II: $x' - y' < x_N^* - y_N^*$. Let $z = \min\{n \in \{0, \dots, N\} : x_n^* - y_n^* = x_N^* - y_N^*\}$. By Lemma 4, $(x_n^*, y_n^*) = (x_N^*, y_N^*)$ for every $n \geq z$.

If $z = 0$, let $(\bar{x}, \bar{y}) = (x', y')$. If $z > 0$, then $x_{z-1}^* - y_{z-1}^* \neq x_N^* - y_N^*$; since d^* is feasible in (OPT2), we must have $x_{z-1}^* - y_{z-1}^* < x_N^* - y_N^*$. Then there exists $\alpha \in [0, 1)$ such that $\alpha(x_N^* - y_N^*) + (1 - \alpha)(x' - y') \geq x_{z-1}^* - y_{z-1}^*$; let $(\bar{x}, \bar{y}) = (\alpha x_N^* + (1 - \alpha)x', \alpha y_N^* + (1 - \alpha)y')$. Since $W_b(\cdot, \cdot, \theta_N)$ is affine and $U_b(\cdot, \cdot, \theta_N)$ is concave in (x, y) , and since $W_b(x', y', \theta_N) > W_b(x_N^*, y_N^*, \theta_N)$, we then have $W_b(\bar{x}, \bar{y}, \theta_N) > W_b(x_N^*, y_N^*, \theta_N)$ and $U_b(\bar{x}, \bar{y}, \theta_N) \geq U_b(x_N^*, y_N^*, \theta_N)$.

Then for every $n \geq z$, we have

$$\begin{aligned} U_b(\bar{x}, \bar{y}, \theta_n) &\geq U_b(x_N^*, y_N^*, \theta_n) - u(\theta_n - \theta_n)((\bar{x} - \bar{y}) - (x_N^* - y_N^*)) \\ &\quad + U_b(\bar{x}, \bar{y}, \theta_n) - U_b(x_N^*, y_N^*, \theta_n) \\ &\geq U_b(x_N^*, y_N^*, \theta_n) = U_b(x_N^*, y_N^*, \theta_z) + u(\theta_n - \theta_z)(x_N^* - y_N^*) \\ &\geq U_b(x_N^*, y_N^*, \theta_z) + u(\theta_n - \theta_z)(\bar{x} - \bar{y}) \geq 0, \end{aligned}$$

where nonnegativity holds since $0 < x' - y' < \bar{x} - \bar{y}$, and since $U_b(x_N^*, y_N^*, \theta_z) \geq 0$ by feasibility of d^* in (OPT2).

Now for each $n \geq z$, apply Lemma 5 to (\bar{x}, \bar{y}) with $\alpha = \theta_n$, $h_1 = \theta_n$, $h_0 = 1 - \theta_n$, and $y = U_b(x_N^*, y_N^*, \theta_z) + u(\theta_n - \theta_z)(\bar{x} - \bar{y})$ to obtain $(\tilde{x}_n, \tilde{y}_n)$ such that $\tilde{x}_n - \tilde{y}_n = \bar{x} - \bar{y}$, $W_b(\tilde{x}_n, \tilde{y}_n, \theta_n) \geq W_b(\bar{x}, \bar{y}, \theta_n)$, and

$$\begin{aligned} U_b(\tilde{x}_n, \tilde{y}_n, \theta_n) &= U_b(x_N^*, y_N^*, \theta_z) + u(\theta_n - \theta_z)(\bar{x} - \bar{y}) \\ &= u \sum_{i < z} (\theta_{i+1} - \theta_i)(x_i^* - y_i^*) + u \sum_{i=z}^{n-1} (\theta_{i+1} - \theta_i)(\tilde{x}_i - \tilde{y}_i), \end{aligned}$$

where the second equation holds since d^* is feasible in (OPT2). Then $\tilde{d} = \{(x_n^*, y_n^*)\}_{n=0}^{z-1} \cup \{(\tilde{x}_n, \tilde{y}_n)\}_{n=z}^N$ is feasible in (OPT2): For each $n > z$, $\tilde{x}_n - \tilde{y}_n = \bar{x} - \bar{y} = \tilde{x}_{n-1} - \tilde{y}_{n-1}$. If $z > 0$, then $\tilde{x}_z - \tilde{y}_z = \bar{x} - \bar{y} \geq x_{z-1}^* - y_{z-1}^*$ by construction, while if $z = 0$, then $\tilde{x}_0 - \tilde{y}_0 = x' - y' > 0$ since $W_b(x', y', \theta_N) > W_b(x_N^*, y_N^*, \theta_N) \geq 0$.

Moreover, since $W_b(\bar{x}, \bar{y}, \theta_N) > W_b(x_N^*, y_N^*, \theta_N)$ and $\theta_z \leq \theta_N < b$, we have

$$0 < (\bar{y} - y_N^*)(\theta_N w_1 + (1 - \theta_N)w_0) + \theta_N w_1((\bar{x} - \bar{y}) - (x_N^* - y_N^*)).$$

Since $\bar{x} - \bar{y} < x_N^* - y_N^*$, this implies $\bar{y} < y_N^*$. Therefore

$$w_1(\bar{x} - x_N^*) - w_0(\bar{y} - y_N^*) = w_1((\bar{x} - \bar{y}) - (x_N^* - y_N^*)) + (w_1 - w_0)(\bar{y} - y_N^*) < 0.$$

Thus, for every $n \geq z$,

$$\begin{aligned} W_b(\bar{x}, \bar{y}, \theta_n) - W_b(x_N^*, y_N^*, \theta_n) &= W_b(\bar{x}, \bar{y}, \theta_n) - W_b(x_N^*, y_N^*, \theta_N) \\ &\quad + (\theta_n - \theta_N)(w_1(\bar{x} - x_N^*) - w_0(\bar{y} - y_N^*)) > 0; \\ \Rightarrow W_b(\tilde{x}_n, \tilde{y}_n, \theta_n) &\geq W_b(\bar{x}, \bar{y}, \theta_n) > W_b(x_N^*, y_N^*, \theta_n). \end{aligned}$$

Then \tilde{d} achieves a strictly higher value of the objective in (OPT2) than d^* , a contradiction.

□

References

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