Online Appendix for "Matching with Strategic Consistency"[*]

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S.1 Strategic Consistency and Bargaining Power: Example 3 Revisited

Suppose we want use Theorem 3 to construct the Pareto-optimal profiles for which the outcomes found in Example 3 are stable. To do so, we must first assign cardinal values to the agents' payoffs. Suppose that each agent receives payoff e^2 from their most preferred outcome, e from their second most preferred outcome, and e^{-10} from outcomes that they regard as worse than autarky.[1] We consider two social welfare functions and contrast the profiles they produce. First, consider the Nash product $\phi(x) = \prod_{i \in I} x_i^{\alpha_i}$ with asymmetric weights $(\alpha_a, \alpha_b, \alpha_1, \alpha_2) = (0.3, 0.2, 0.35, 0.15)$. Then in the profile $\{C_i^\phi, \mu_i^\phi\}_{i \in I}$, (4) tells us that the agents choose (and believe others will choose) the contracts in the nonstrategically individually rational outcome that yields the highest value of ϕ among all such outcomes that are subsets of the set of available contracts: i.e., the algorithm (3) can construct $\{C_i^\phi, \mu_i^\phi\}_{i \in I}$

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[1] Formally, we let $u_i(\emptyset) = 1$ for each $i \in I$, and

$$
\begin{aligned}
&u_a(\{x_{a1}\}) = e^2; && u_a(\{x_{a2}\}) = e; && u_a(\{x_{a1}, x_{a2}\}) = e^{-10};\\
&u_b(\{x_{b2}\}) = e^2; && u_b(\{x_{b1}\}) = e; && u_b(\{x_{b1}, x_{b2}\}) = e^{-10};\\
&u_1(\{x_{a1}\}) = e^{-10}; && u_1(\{x_{b1}\}) = e; && u_1(\{x_{a1}, x_{b1}\}) = e^2;\\
&u_2(\{x_{a2}\}) = e^2; && u_2(\{x_{b2}\}) = e; && u_2(\{x_{a2}, x_{b2}\}) = e^{-10}.
\end{aligned}
$$

using the order $\succ^{\phi}$ given by

$$\{x_{a1}, x_{b1}\} \succ^{\phi} \{x_{a2}, x_{b1}\} \succ^{\phi} \{x_{b2}\} \succ^{\phi} \{x_{a2}\} \succ^{\phi} \{x_{b1}\} \succ^{\phi} \emptyset,$$

and so the stable outcome for $\{C_i^{\phi}, \mu_i^{\phi}\}_{i \in I}$ is given by $\mu^{\phi}(X) = \{x_{a1}, x_{b1}\}$.[2]

Now suppose that we change the weights to be more favorable to firm 2: Let $(\alpha'_a, \alpha'_b, \alpha'_1, \alpha'_2) = (0.3, 0.2, 0.15, 0.35)$, and $\phi'(x) = \prod_{i \in I} x_i^{\alpha'_i}$. Then, the values taken by the social welfare function become higher at outcomes where firm 2 hires Bob relative to those where it hires no one, and at outcomes where firm 2 hires Alice relative to those where it hires Bob: the ordering $\succ^{\phi'}$ is

$$\{x_{a2}, x_{b1}\} \succ^{\phi'} \{x_{a1}, x_{b1}\} \succ^{\phi'} \{x_{a2}\} \succ^{\phi'} \{x_{b2}\} \succ^{\phi'} \{x_{b1}\} \succ^{\phi'} \emptyset,$$

and the stable outcome for $\{C_i^{\phi'}, \mu_i^{\phi'}\}_{i \in I}$ is $\mu^{\phi'}(X) = \{x_{a2}, x_{b1}\}$. ■

S.2 Other Stability Concepts

While the matching literature generally focuses on the solution concept — stability — that we adopt in this paper, many papers also consider other related matching-theoretic solution concepts. Two of the most common are *setwise stability* (Sotomayor, 1999) and *weak setwise stability* (Klaus and Walzl, 2009).

These solution concepts make the same predictions in two-sided one-to-one matching markets (Echenique and Oviedo, 2006; Klaus and Walzl, 2009). But in two-sided many-to-many matching markets (Echenique and Oviedo, 2006; Klaus and Walzl, 2009) or environments with multilateral contracts (Bando and Hirai, 2021), there is, in general, a gap between them. Because it requires that agents' beliefs must be correct, strategic consistency closes this gap between the predictions of stability and weak setwise stability: Whenever a block (in the sense used in stability) is successful, each agent must agree on the set of contracts that will be chosen (Lemma 1). Since the difference between stability and weak setwise stability is that the latter only considers blocks with this property, the two solution concepts must coincide.[3] Hence,

[2] The log values taken by the two social welfare functions ϕ and ϕ' at the nonstrategically individually rational outcomes are given by

$$\begin{aligned}
&\log\phi((u_i(\emptyset))_{i\in I}) = 0; && \log\phi((u_i(\{x_{a2}\}))_{i\in I}) = 0.6; && \log\phi((u_i(\{x_{a2}, x_{b1}\}))_{i\in I}) = 1.15;\\
&\log\phi((u_i(\{x_{b1}\}))_{i\in I}) = 0.55; && \log\phi((u_i(\{x_{b2}\}))_{i\in I}) = 0.7; && \log\phi((u_i(\{x_{a1}, x_{b1}\}))_{i\in I}) = 1.5,\\
&\log\phi'((u_i(\emptyset))_{i\in I}) = 0; && \log\phi'((u_i(\{x_{a2}\}))_{i\in I}) = 1; && \log\phi'((u_i(\{x_{a2}, x_{b1}\}))_{i\in I}) = 1.35;\\
&\log\phi'((u_i(\{x_{b1}\}))_{i\in I}) = 0.35; && \log\phi'((u_i(\{x_{b2}\}))_{i\in I}) = 0.75; && \log\phi'((u_i(\{x_{a1}, x_{b1}\}))_{i\in I}) = 1.1.
\end{aligned}$$

[3] In the literature, weak setwise stability is defined for environments without externalities, but it can be extended to accommodate externalities in the same way as we extended stability. Formally, a set of contracts Y is weakly setwise stable if it is individually rational and there is no $Z \subseteq X$ such that $Z_i = C_i(Z_i \cup Y_i | Z_{-i} \cup Y_{-i})$ for each $i \in N(Z \setminus Y)$.

strategic consistency allows stability to capture outcomes that, with nonstrategic choice, are only captured by weak setwise stability. Example S.1 illustrates.

Example S.1 (Strategic Consistency vs. Nonstrategic Choice). Consider a market with two agents $I = \{1, 2\}$ and two contracts $X = \{x, z\}$ they can sign with each other, and suppose that they have preferences

$$u_1(\{x,z\}) > u_1(\{z\}) > u_1(\{x\}) > u_1(\emptyset); \qquad u_2(\{x\}) > u_2(\{z\}) > u_2(\emptyset) > u_2(\{x,z\}).$$

This is a two-sided market where contracts are substitutable, so with nonstrategic choice, there is a (unique) stable outcome, $\{x\}$. But another outcome is also compelling. Observe that $\{z\}$ is blocked by $\{x\}$ when choice is nonstrategic, since $\hat{C}_1(\{x,z\}) = \{x,z\}$ and $\hat{C}_2(\{x,z\}) = \{x\}$. But this block only occurs because agent 1 incorrectly assumes that agent 2 will choose z as well as x. With strategic consistency, however, the agents must agree: either that the block will result in z being dropped (in profiles where $\{x\}$ is stable) or that it will lead both agents to choose z alone (in profiles where $\{z\}$ is stable). This allows strategic consistency to identify both outcomes as consistent with stability, whereas nonstrategic choice identifies only one. ■

S.3 Other Refinements

S.3.1 Forward Induction

Even though they are correct and cross-set consistent, beliefs may make predictions about others' responses to proposed cooperative deviations that are not justified once the credibility of those deviations is taken into account. This section refines the behavior captured by the set of strategically consistent profiles to rule out such beliefs using a notion of credibility based on forward induction: a blocking proposal is credible if no one has a reason to make the proposal without intending to follow through on it.[4] Moreover, the forward induction reasoning that it captures is *farsighted*: If an agent believes that a deviation is credible, it must not enlarge the set of outcomes that can be reached through a sequence of further deviations that are consistent with the agents' beliefs. As Example 4 shows, when we rule out profiles that are not robust to such credible deviations, strategic consistency allows stability to make intuitive predictions in settings where nonstrategic choice does not yield a stable outcome — and in the network formation context, those where no pairwise stable outcome exists (e.g., the formation of a trading network in Jackson and Watts (2002)).

[4]In other words, a credible blocking proposal is one that no agent would choose to propose *unless their intentions in doing so, if fully understood by others, would prompt a response that justifies that deviation.* This is analogous to the kind of credible deviation discussed in, e.g., Kohlberg and Mertens (1986): one that an agent would not choose to make *unless their intentions in doing so, if fully understood by others, would prompt a response that justifies that deviation.*

Formally, given a profile of choice functions and beliefs $\{C_i, \mu_i\}_{i \in I}$, we say that the sequence of outcomes $\{Z^n\}_{n=0}^N$ is a *farsighted chain from Z to Z'* if $Z^0 = Z$; $Z^N = Z'$; and for each n and each $i \in I$, $\mu_i(Z^n \cup Z^{n+1}) = Z^{n+1}$. In words, a farsighted path is a sequence of outcomes where each agent believes the $n+1$st outcome will result from a block of the nth.

Definition (Credible Blocks and Forward Induction). Given a strategically consistent profile $\{C_i, \mu_i\}_{i \in I}$, we say that Z is a *credible blocking proposal* for Y if

i. (Nonstrategic Individual Rationality) Z is nonstrategically individually rational;

ii. (Myopic Credibility) even if agents myopically believe that others will leave their existing contracts intact, each will agree to the new contracts in Z, and at least one will reject each of the old contracts in $Y \setminus Z$: $Z = \bigcap_{i \in I}(\hat{C}_i((Y \cup Z)_i | (Y \cup Z)_{-i}) \cup (Y \cup Z)_{-i})$;[5] and

iii. (Farsighted Credibility) adopting the new set of contracts cannot lead to new deviations *farsightedly*: If there is a farsighted chain from Z to Z', there is a farsighted chain from Y to Z'.

We say that $\{C_i, \mu_i\}_{i \in I}$ satisfies *forward induction* if for every nonstrategically individually rational outcome Y, and every Z that is a credible blocking proposal for Y, we have $\mu_i(Y \cup Z) = Z$ for each $i \in I$. If this property holds whenever $Y \subset Z$, we say that $\{C_i, \mu_i\}_{i \in I}$ satisfies *weak forward induction.*

Blocking proposals are credible if no agent has a reason to make them unless they intend to enact the newly proposed set of contracts. In particular, they cannot benefit *directly* by misleading others about their intentions, and either keeping old contracts or rejecting the new ones that are part of the blocking proposal (myopic credibility); and they cannot benefit *indirectly* by subsequently deviating unilaterally (nonstrategic individual rationality) or jointly (farsighted credibility).

Weak forward induction requires each agent to believe that the members of a blocking coalition will go along with credible proposals to *add* contracts. Forward induction requires agents to also believe in credible proposals to *change* the set of contracts, i.e., add some contracts while deleting others.[6]

[5] That is, with nonstrategic choice, Z results from a block of Y.

[6] To motivate weak forward induction, note that it is more difficult for agents to evaluate the credibility of proposals to change the set of contracts than the credibility of proposals that only add contracts: If a proposal changes the set of contracts, Evaluating myopic credibility requires determining not just whether agents in the blocking coalition can benefit from rejecting newly proposed contracts, but also whether they might benefit from keeping their *existing* contracts. Moreover, unlike with proposals to add contracts, the credibility of a proposal to change the set of contracts does not follow from its nonstrategic individual rationality (Lemma S.2).

In general, our forward induction refinements are neither stronger nor weaker than Pareto optimality. But Pareto-optimal profiles must satisfy forward induction when externalities are absent, since in that case, any myopically credible blocking proposal is also a Pareto improvement.

Lemma S.1 (Pareto Optimality and Forward Induction). *If there are no externalities, and* $\{C_i, \mu_i\}_{i \in I}$ *is strategically consistent and satisfies Pareto optimality, it also satisfies forward induction.*

More generally, Theorem S.1 shows that weak forward induction significantly refines the set of strategically consistent profiles: we can rule out all profiles from Proposition **??** except those whose stable outcomes are *maximal*. Moreover, strategically consistent profiles that satisfy weak forward induction always exist.

Theorem S.1 (Weak Forward Induction: Existence and Characterization).

i. Strategically consistent profiles that satisfy weak forward induction exist.

ii. There is a strategically consistent profile satisfying weak forward induction for which $S \subseteq X$ *is stable if and only if* S *is nonstrategically individually rational and there is no* $Z \supset S$ *that is nonstrategically individually rational.*

Like in Theorems 2-4, the nonstrategically individually rational outcomes play an important role in Theorem S.1. Because of weak forward induction, however, agents must not only believe that others will choose a nonstrategically individually rational subset from any set of available contracts Z, but a *maximal* one: That is, there can be no nonstrategically individually rational Y such that $Z \supseteq Y \supset \mu(Z)$. This is because, for blocking proposals that *add* contracts, credibility follows from nonstrategic individual rationality (Lemma S.2 in the appendix).[7]

Our algorithm (3) generates beliefs with this maximality property precisely when the order used to initialize it is structured so that it ranks larger outcomes higher than smaller ones.[8] Since the resulting profile's unique stable outcome is the one that the order ranks highest, it must be maximal among all nonstrategically individually rational outcomes.

Identifying profiles that satisfy the full forward induction criterion is more challenging, precisely because it is more difficult to evaluate the credibility of blocks when they not only

[7]If $Z \supset Y$ is nonstrategically individually rational, it is by definition a *myopically* credible deviation from Y. Moreover, cross-set consistency ensures that $\mu_i(Z \cup S) = S \Rightarrow \mu_i(Y \cup S) = S$, and correctness and optimality ensure that choices match beliefs (Lemma 1), so any farsighted chain starting at $Z^0 = Z$ can be changed into a farsighted chain starting at Y merely by letting $Z^0 = Y$ (farsighted credibility).

[8]If it is initialized with an order wtihout this monotonicity property, some nonstrategically individually rational outcome must be ranked below one of its subsets, and so when it is available, agents will choose — and correctly believe others will choose — only the contracts from the higher ranked subset.

add new contracts, but delete existing ones. One condition that allows us to do so ensures that any beliefs that are part of a strategically consistent profile can make pairwise comparisons between nonstrategically individually rational outcomes Y, Z: i.e., either $\mu_i(Y \cup Z) = Y$ or $\mu_i(Y \cup Z) = Z$. Formally, we say that the set of outcomes $\mathcal{M} \subseteq 2^X$ is *pairwise comparable* if, for any $S, Y, Z \in \mathcal{M}$ such that $S \subseteq Y \cup Z$, either $S \subseteq Y$ or $S \subseteq Z$.

Theorem S.2 (Forward Induction: Existence). *If the set of nonstrategically individually rational outcomes is pairwise comparable, a strategically consistent profile satisfying forward induction exists.*

The profile in Theorem S.2 is once again constructed using our algorithm (3). To ensure that it satisfies forward induction, we initialize our algorithm with an order that not only ranks larger outcomes higher, but ensures that whenever an outcome Y is ranked above a myopically credible blocking proposal Z for Y, there is some other outcome ranked in between.[9]

When the order $\succ$ has this structure, our algorithm always generates a profile in which each agent believes that the others will go along with *all* credible blocking proposals, even those that remove existing contracts, because those proposals are always higher-ranked than the outcomes they block. In particular, given any outcome Y, any lower-ranked outcome Z is either not a farsightedly credible blocking proposal for Y (if there are outcomes ranked in between) or not a myopically credible blocking proposal for Y (otherwise).[10]

S.3.2 Nonstrategic Robustness

Even when they satisfy forward induction, some strategically consistent profiles may pin down stable outcomes that are *not* stable when choice is nonstrategic. This is a double-edged sword: On the one hand, this is the feature of strategic consistency that allows us to apply stability to environments where the standard, nonstrategic approach leads to nonexistence. On the other hand, outcomes that are *also* robust to nonstrategic deviations, not just deviations based on correct beliefs, may be more appealing.

Hence, we propose a simple refinement that selects profiles whose stable outcomes are robust in this way. Formally:

Definition (Nonstrategic Robustness). We say that a strategically consistent profile $\{C_i, \mu_i\}_{i \in I}$ yields a *nonstrategically robust* stable outcome if its unique stable outcome $\mu(X)$ is stable given the profile of nonstrategic choice functions $\{\hat{C}_i\}_{i \in I}$.

[9] The proof focuses on showing that such an order exists.

[10] When there are outcomes ranked between Z and Y, Z cannot be a farsightedly credible blocking proposal for Y: Since the nonstrategically individually rational outcomes are pairwise comparable, there is a farsighted path from Z, but not from Y, to any outcome ranked between them.

Our key result about this refinement is that it refines weak forward induction; that is, strategic consistency and weak forward induction never overturn the predictions of the standard approach to choice.

Theorem S.3 (Nonstrategic Robustness and Weak Forward Induction). *If an outcome is stable given the profile of nonstrategic choice functions $\{\hat{C}_i\}_{i\in I}$, it is the nonstrategically robust stable outcome of some strategically consistent assessment satisfying weak forward induction.*

Intuitively, if an outcome is stable given nonstrategic choice functions, it is, by definition, nonstrategically individually rational. Moreover, there cannot be a myopically credible blocking proposal to add contracts to it, as there must be if there was a *larger* nonstrategically individually rational outcome. Hence, it must be one of the outcomes characterized in Theorem S.1.

S.3.3 Proofs for Appendix S.3

Proof of Lemma S.1 (Pareto Optimality and Forward Induction) Suppose that given $\{C_i, \mu_i\}_{i\in I}$, Z is a credible blocking proposal for Y. Then since there are no externalities, myopic credibility implies $u_i(Z_i) \geq u_i(S_i)$ for each $S \subseteq Y \cup Z$. By assumption, for any such $S \subseteq Y \cup Z$, $u_i(S_i) \neq u_i(Z_i)$, and hence $u_i(S_i) > u_i(Z_i)$, for each $i \in I$ such that $S_i \neq Z_i$. Then since $\{C_i, \mu_i\}_{i\in I}$ satisfies Pareto optimality, and Z is nonstrategically individually rational, we have $\mu_i(Z \cup S) \neq S$ for each $S \subseteq Y \cup Z$ with $S \neq Z$ and each $i \in I$. Then by cross-set consistency, we have $\mu_i(Z \cup Y) \neq S$ for each $S \subseteq Y \cup Z$ with $S \neq Z$ and each $i \in I$. Then by elimination, we must have $\mu_i(Z \cup Y) = Z$ for each $i \in I$. Thus, $\{C_i, \mu_i\}_{i\in I}$ satisfies forward induction. □

Lemma S.2. *Given a strategically consistent profile $\{C_i, \mu_i\}_{i\in I}$, $Z \supset Y$ is a credible blocking proposal for Y if and only if it is nonstrategically individually rational.*

Proof. (If) Suppose $Z \supset Y$ is nonstrategically individually rational.

Z is a myopically credible blocking proposal for Y: Since $Z \supset Y$ and Z is nonstrategically individually rational,

$$\bigcap_{i\in I}(\hat{C}_i((Y \cup Z)_i|(Y \cup Z)_{-i}) \cup (Y \cup Z)_{-i}) = \bigcap_{i\in I}(\hat{C}_i(Z_i|Z_{-i}) \cup Z_{-i}) = \bigcap_{i\in I}(Z_i \cup Z_{-i}) = Z.$$

Z is a farsightedly credible blocking proposal for Y: Suppose there is a farsighted chain $\{Z^n\}_{n=0}^N$ from Z to Z'. Then for each $i \in I$, $\mu_i(Z \cup Z^1) = Z^1$. Then since $\{C_i, \mu_i\}_{i\in I}$ is strategically consistent and $Z \supset Y$, by Lemma 4, $\mu_i(Y \cup Z^1) = Z^1$. Then $\{Y, \{Z^n\}_{n=1}^N\}$ is a farsighted chain from Y to Z'.

Then since Z is nonstrategically individually rational, it is a credible blocking proposal for Y.

(Only if) Suppose $Z \supset Y$ is a credible blocking proposal for Y. Then by definition, it is nonstrategically individually rational. $\square$

Corollary S.1. *A strategically consistent profile $\{C_i, \mu_i\}_{i \in I}$ satisfies weak forward induction if and only if $\mu_i(Y) = \mu(Y) = Y$ for each $i \in I$ whenever Y is nonstrategically individually rational.*

Proof. (Only if) Suppose $\{C_i, \mu_i\}_{i \in I}$ is strategically consistent and satisfies weak forward induction, and that Y is nonstrategically individually rational. If $Y = \emptyset$, then $C_i(Y_i|Y_{-i}) = \emptyset$ for each $i \in I$; then since beliefs are correct, $\mu_i(Y) = \emptyset = Y$ for each $i \in I$. Alternatively, if $Y \neq \emptyset$, then by Lemma S.2, Y is a credible blocking proposal for $\emptyset$. Moreover, $\emptyset$ is nonstrategically individually rational, since by definition, $\hat{C}_i(\emptyset|\emptyset) = \emptyset$ for each $i \in I$. Then since $\{C_i, \mu_i\}_{i \in I}$ satisfies weak forward induction, we must have $\mu_i(Y) = \mu_i(Y \cup \emptyset) = Y$ for each $i \in I$.

(If) Suppose $\{C_i, \mu_i\}_{i \in I}$ is strategically consistent and that $\mu_i(Y) = Y$ for each nonstrategically individually rational $Y \subseteq X$ and each $i \in I$. Consider $Y, Z \subseteq X$ such that Z is a credible blocking proposal for Y, and $Z \supset Y$. By definition, Z is nonstrategically individually rational; then $\mu_i(Y \cup Z) = \mu_i(Z) = Z$. Then $\{C_i, \mu_i\}_{i \in I}$ satisfies weak forward induction. $\square$

Proof of Theorem S.1 (Weak Forward Induction: Existence and Characterization) (ii): (Only if) Suppose that S is stable for the strategically consistent profile $\{C_i, \mu_i\}_{i \in I}$, and that $\{C_i, \mu_i\}_{i \in I}$ satisfies weak forward induction. Then by Lemma 7, S is nonstrategically individually rational. Now suppose toward a contradiction that there exists $Z \supset S$ that is also nonstrategically individually rational. Then by Corollary S.1, $\mu_i(Z) = Z$ for each $i \in I$, since $\{C_i, \mu_i\}_{i \in I}$ satisfies weak forward induction. Since S is stable for $\{C_i, \mu_i\}_{i \in I}$, by Corollary 1, $\mu_i(X) = S$ for each $i \in I$. Then since $\{C_i, \mu_i\}_{i \in I}$ is strategically consistent, by Lemma 4, $\mu_i(Z) = S$ for each $i \in I$, a contradiction.

(If) Suppose that S is nonstrategically individually rational and there is no $Z \supset S$ that is nonstrategically individually rational. To construct a strategically consistent profile $\{C_i, \mu_i\}_{i \in I}$ satisfying WFI for which S is stable, we first define an order $\succ$ that refines the subset order $\supset$ and ranks S highest, and then use it in our algorithm (3) and show that the resulting profile satisfies weak forward induction.

Let $\mathcal{M} = \{Z \subseteq X | \hat{C}_i(Z_i|Z_{-i}) = Z_i \text{ for each } i \in I\}$ denote the set of nonstrategically individually rational outcomes, and label its elements according to the sequence $\{Y^n\}_{n=1}^{|\mathcal{M}|}$, constructed inductively as follows: For the initial element, choose $Y^1 = S$. Then, given elements $\{Y^n\}_{n=1}^m$, choose $Y^{m+1} \in \mathcal{M} \setminus \{Y^n\}_{n=1}^m$ such that there is no $Y' \in \mathcal{M} \setminus \{Y^n\}_{n=1}^m$ with $Y' \supset Y^{m+1}$. This construction implies that whenever $Y^n \supset Y^m$, we must have $n < m$: If

$n > m$, then $Y^n \in \mathcal{M} \setminus \{Y^k\}_{k=1}^{m-1}$, and so Y^m could not have been chosen as the mth element of the sequence.

Now define the order $\succ$ on $\mathcal{M}$ as follows: $Y^n \succ Y^m \Leftrightarrow n < m$. This order refines $\supset$ on $\mathcal{M}$: If $Y^n \supset Y^m$, then $n < m$, and hence $Y^n \succ Y^m$.

Let $\{C_i, \mu_i\}_{i \in I}$ be the profile of choice functions and beliefs constructed according to (3). By Lemma 6, $\{C_i, \mu_i\}_{i \in I}$ is strategically consistent. Since $S = Y^1$, we have $S \succeq Y$ for all $Y \in \mathcal{M}$, and hence $S = \max_{\succ}\{Y' | Y' \subseteq X\} = \mu_i(X)$ for each $i \in I$. Then by Corollary 1, $\mu(X) = S$ is uniquely stable for $\{C_i, \mu_i\}_{i \in I}$.

Now we show that $\{C_i, \mu_i\}_{i \in I}$ satisfies weak forward induction. Since $\succ$ refines $\supset$ on $\mathcal{M}$, for any $Y, Y' \in \mathcal{M}$, $Y \supset Y' \Rightarrow Y \succ Y'$. Then by (3), if $Y \in \mathcal{M}$, $\mu_i(Y) = \max_{\succ}\{Y' | Y' \subseteq Y\} = Y$ for each $i \in I$. It follows from Corollary S.1 that $\{C_i, \mu_i\}_{i \in I}$ satisfies weak forward induction.

(i): The set $\mathcal{M}$ of nonstrategically individually rational outcomes is nonempty, since it always contains the autarky outcome $\emptyset$. (By definition, we must have $\hat{C}_i(\emptyset|\emptyset) = \emptyset$ for all $i \in I$.) Moreover, since X is finite, so is 2^X, and so $\mathcal{M} \subseteq 2^X$ is finite as well. It follows straightforwardly that $\mathcal{M}$ must contain at least one element S such that there is no $Z \in \mathcal{M}$ with $Z \supset S$. □

The proof of Theorem S.2 relies on techniques from graph theory.[11] Consider a directed graph which has an edge from Z to Y whenever Z is *not* a myopically credible blocking proposal for Y. Then, consider the set of paths on this graph that (a) are $\subseteq$-nondecreasing, i.e., never pass through Z after $Y \supset Z$, and (b) do not pass through any nodes more than once. If we choose one of the longest of these paths $\{Y^n\}_{n=1}^N$, it must pass through each node: Any node Y the path does not pass through can be inserted somewhere along the path that is after each of its subsets $Z \subset Y$ and before each of its supersets $S \supset Y$.

Then we can construct the profile in Theorem S.2 the same way (3) as in Theorem 2, but this time ordering the nonstrategically individually rational outcomes by their position along the path: $Y \succ Z \Leftrightarrow Y = Y^n$, $Z = Y^m$, for $n > m$. Because of pairwise comparability and the $\subseteq$-nondecreasing nature of the path, agents always believe that a higher-ranked outcome will result from a block of a lower-ranked outcome: $Y \succ Z \Rightarrow \mu(Y \cup Z) = Y$.

Consequently, forward induction only requires that lower-ranked outcomes Y^m are not credible blocking proposals for higher-ranked ones Y^n. If the higher-ranked one is the direct successor of the lower-ranked one — i.e., if $n = m + 1$ — then by the path's construction, it is not a myopically credible blocking proposal. Otherwise, if $n > m + 1$, there is some outcome Y^k ranked between the other two — i.e., with $n > k > m$ — which is (by construction) at the end of a farsighted chain from the lower-ranked outcome, but not the higher-ranked one.

Proof of Theorem S.2 (Forward Induction: Existence)

[11] We thank Akhil Vohra for the idea for this construction.

Step 0: The myopically non-credible order $\trianglerighteq$. Let $\mathcal{M} = \{Z \subseteq X | \hat{C}_i(Z_i|Z_{-i}) = Z_i \text{ for each } i \in I\}$ denote the set of nonstrategically individually rational outcomes, and define an order $\trianglerighteq$ on $\mathcal{M}$ as follows: $Y \trianglerighteq Z \Leftrightarrow Z$ is *not* a myopically credible blocking proposal for Y (i.e., $\bigcap_{i\in I}(\hat{C}_i((Y \cup Z)_i|(Y \cup Z)_{-i}) \cup (Y \cup Z)_{-i}) \neq Z$) or $Z = Y$.

$\trianglerighteq$ is complete, since we cannot have $Z = \bigcap_{i\in I}(\hat{C}_i((Y \cup Z)_i|(Y \cup Z)_{-i}) \cup (Y \cup Z)_{-i}) = Y$ when $Y \neq Z$. Thus, $Y \ntrianglerighteq Z \Rightarrow Y \triangleleft Z$.

Moreover, by Lemma S.2, $\triangleright$ refines $\supset$: $Z \supset Y \Rightarrow Z \triangleright Y$. Hence, by contrapositive, $Z \trianglelefteq Y \Rightarrow Z \not\supset Y$.

Step 1: Choose a longest $\supset$-nondecreasing, nonrepeating, $\trianglerighteq$-successor path $\{Z^n\}_{n=1}^M$. Let $\mathcal{Y}$ be the set of sequences $\{Y^n\}_{n=1}^N \subseteq \mathcal{M}$ such that (a) $n < m$ whenever $Y^n \supset Y^m$, (b) $Y^n \neq Y^m$ whenever $n \neq m$, and (c) for each n, $Y^n \trianglerighteq Y^{n+1}$. Since X is finite, so is 2^X, and hence $\mathcal{M}$. From property (b), any sequence in $\mathcal{Y}$ can have at most $|\mathcal{M}|$ elements from the finite set $\mathcal{M}$, so $\mathcal{Y}$ is finite as well. Then it must have a longest element $\{Z^n\}_{n=1}^M$ such that for any $\{Y^n\}_{n=1}^N \in \mathcal{Y}$, $N \leq M$.

Step 2: The path terminates at $Z^M = \emptyset$. $\emptyset \subseteq Z^n$ for all n, so by condition (a), either $\emptyset = Z^M$ or $\emptyset \notin \{Z^n\}_{n=1}^M$. Suppose toward a contradiction that the latter is true. Then we can append $\emptyset$ to $\{Z^n\}_{n=1}^M$ to create a longer sequence $\{\{Z^n\}_{n=1}^M, \emptyset\}$ that is still part of $\mathcal{Y}$: (a) holds since it holds for $\{Z^n\}_{n=1}^M$, (b) holds since it holds for $\{Z^n\}_{n=1}^M$ and $\emptyset \notin \{Z^n\}_{n=1}^M$, and (c) holds since it holds for $\{Z^n\}_{n=1}^M$ and (since $\triangleright$ refines $\supset$) $Z^M \supset \emptyset \Rightarrow Z^M \triangleright \emptyset$. But $\{Z^n\}_{n=1}^M$ is the longest sequence in $\mathcal{Y}$, a contradiction.

Step 3: The path covers all of $\mathcal{M}$: For each $Y \in \mathcal{M}$, $Y \in \{Z^n\}_{n=1}^M$. Suppose toward a contradiction that $Y \notin \{Z^n\}_{n=1}^M$, and let $K = \min\{n | Z^n \subset Y\}$. ($\{n | Z^n \subset Y\}$ is nonempty, since $Z^M = \emptyset \subset Y$ by Step 2.) Then $Z^K \subset Y$, and since $\triangleright$ refines $\supset$, $Y \triangleright Z^K$.

By definition, $Z^n \subset Y \Rightarrow n \geq K$. Moreover, $Z^n \supset Y \Rightarrow n < K$: if $Z^n \supset Y$, then $Z^n \supset Z^K$, and so by (a) $n < K$.

We use induction to show that $Y \triangleright Z^n$ (and hence, since $\triangleright$ refines $\supset$, $Y \not\subset Z^n$) for all $n < K$:

- **Initial step: $Y \triangleright Z^{K-1}$.** Suppose toward a contradiction that $Y \ntriangleright Z^{K-1}$. Since $\trianglerighteq$ is complete, $Y \trianglelefteq Z^{K-1}$. Then $\{\{Z^n\}_{n=1}^{K-1}, Y, \{Z^n\}_{n=K}^M\} \in \mathcal{Y}$: (c) is satisfied since it holds for $\{Z^n\}_{n=1}^M$ and $Z^{K-1} \trianglerighteq Y \triangleright Z^K$. (b) holds since it holds for $\{Z^n\}_{n=1}^M$ and $Y \notin \{Z^n\}_{n=1}^M$. Finally, (a) holds since it holds for $\{Z^n\}_{n=1}^M$, and we know that $Z^n \subset Y \Rightarrow n \geq K$, and $Z^n \supset Y \Rightarrow n < K$. But $\{Z^n\}_{n=1}^M$ is the longest sequence in $\mathcal{Y}$, a contradiction.

- **Induction step: for any $t \leq K-1$, $Y \triangleright Z^n$ for all $n \in [t, K-1] \Rightarrow Y \triangleright Z^n$ for all $n \in [t-1, K-1]$.** Suppose that $Y \triangleright Z^n$ for all $n \in [t, K-1]$. Since $\triangleright$ refines $\supset$, it follows that $Z^n \not\supset Y$ for all $n \in [t, K-1]$. Since we have already shown that $Z^n \supset Y \Rightarrow n < K$, it follows that $Z^n \supset Y \Rightarrow n < t$.

  Now suppose toward a contradiction that $Y \ntriangleright Z^{t-1}$. Since $\trianglerighteq$ is complete, $Y \trianglelefteq Z^{t-1}$.

Then $\{\{Z^n\}_{n=1}^{t-1}, Y, \{Z^n\}_{n=t}^{M}\} \in \mathcal{Y}$: (c) is satisfied since it holds for $\{Z^n\}_{n=1}^{M}$ and $Z^{t-1} \trianglerighteq Y \triangleright Z^t$. (b) holds since it holds for $\{Z^n\}_{n=1}^{M}$ and $Y \notin \{Z^n\}_{n=1}^{M}$. Finally, (a) holds since it holds for $\{Z^n\}_{n=1}^{M}$, and we know that $Z^n \subset Y \Rightarrow n \geq t$, and $Z^n \supset Y \Rightarrow n < t$. But $\{Z^n\}_{n=1}^{M}$ is the longest sequence in $\mathcal{Y}$, a contradiction.

Consequently, $Y \triangleright Z^1$, and (since we have already shown that $Z^n \supset Y \Rightarrow n < K$) $Y \not\subset Z^n$ for all n. Since (a) and (c) both hold for $\{Z^n\}_{n=1}^{M}$, it follows that $\{Y, \{Z^n\}_{n=1}^{M}\}$ satisfies (a) and (c). And since $\{Z^n\}_{n=1}^{M}$ satisfies (b) and $Y \notin \{Z^n\}_{n=1}^{M}$, $\{Y, \{Z^n\}_{n=1}^{M}\}$ satisfies (b) as well. Then $\{Y, \{Z^n\}_{n=1}^{M}\} \in \mathcal{Y}$, a contradiction since $\{Z^n\}_{n=1}^{M}$ is the longest sequence in $\mathcal{Y}$.

Step 4: Construction of a strategically consistent profile. By Step 3 and since $\{Z^n\}_{n=1}^{M} \in \mathcal{Y}$, every element of $\mathcal{M}$ appears exactly once in $\{Z^n\}_{n=1}^{M}$. Hence, we can define a new strict total order $\succ$ on $\mathcal{M}$ as follows: $Z^n \succ Z^m \Leftrightarrow n < m$. Since $\{Z^n\}_{n=1}^{M}$ satisfies (c), this order refines $\supset$ on $\mathcal{M}$: If $Z^n \supset Z^m$, then $n < m$, and hence $Z^n \succ Z^m$. Let $\{C_i, \mu_i\}_{i \in I}$ be the profile of choice functions and beliefs constructed according to (3). By Lemma 6, $\{C_i, \mu_i\}_{i \in I}$ is strategically consistent.

Step 5: Common beliefs are rationalized by $\succ$: If $Y \succ Z$, then $\mu_i(Y \cup Z) = Y$ for each $i \in I$. Suppose $Y, Z \in \mathcal{M}$ are such that $Y \succ Z$. Since $\mathcal{M}$ is pairwise comparable, for all $S \subseteq Y \cup Z$, either $S \subseteq Y$ or $S \subseteq Z$. Then since $\succ$ refines $\supset$, for all $S \subseteq Y \cup Z$, either $S \preceq Y$ or $S \preceq Z$, and so by transitivity of $\succ$, $S \prec Y$. It follows that for each $i \in I$, $\mu_i(Y \cup Z) = \max_{\succ}\{Y' | Y' \subseteq Y \cup Z\} = Y$.

Step 6: Farsighted paths are $\succ$-increasing: If $Y \in \mathcal{M}$, and there is a farsighted path from Y to Z, then $Z \succ Y$. Suppose there is a farsighted path $\{Y^n\}_{n=0}^{N}$ from Y to Z. Since $\mu_i(Y^{n-1} \cup Y^n) = Y^n$ for each $i \in I$ and $n > 0$, by construction of μ, $Y^n \in \mathcal{M}$ for each $n > 0$. Since $Y^0 = Y \in \mathcal{M}$ by assumption, it follows from Step 5 that for each n, $Y^{n+1} \not\prec Y^n$. Then since $\succ$ is a strict total order on $\mathcal{M}$, $Y^{n+1} \succ Y^n$ for each $n < N$, and so by transitivity $Z \succ Y$.

Step 7: $\{C_i, \mu_i\}_{i \in I}$ satisfies forward induction. Suppose toward a contradiction that Z is a credible blocking proposal for $Y \in \mathcal{M}$, but $\mu_i(Z \cup Y) \neq Z$ for some $i \in I$. Then by Step 5, $Z \not\succ Y$. By definition, $Z \in \mathcal{M}$; since, by construction, $\succ$ is a strict total order on $\mathcal{M}$, we have $Z \prec Y$. Moreover, since Z is a myopically credible blocking proposal for Y, and $\trianglerighteq$ is complete, $Z \triangleright Y$.

Since $Y, Z \in \mathcal{M}$, by Step 3, they must be elements of the sequence $\{Z^n\}_{n=1}^{M}$; label $Y = Z^y$ and $Z = Z^z$. By definition of $\succ$, since $Y \succ Z$, we must have $y < z$. Then since $Z^z \triangleright Z^y$, and $\{Z^n\}_{n=1}^{M}$ satisfies property (c), we must have $y < z - 1$. Then $Y \succ Z^{z-1} \succ Z$, and so (by Step 5) there is a farsighted path from Z to Z^{z-1}, but (by Step 6) there is no farsighted path from Y to Z^{z-1}. Then Z is not a farsightedly credible blocking proposal for Y, a contradiction. □

Proof of Theorem S.3 Suppose Y is stable given $\{\hat{C}_i\}_{i \in I}$. Then by definition, it is nonstrategically individually rational: $\hat{C}_i(Y_i|Y_{-i}) = Y_i$ for each $i \in I$. Moreover, there is no $Y' \supset Y$ such that $\hat{C}_i(Y'_i|Y'_{-i}) = Y'_i$ for each $i \in I$: Suppose not, and there exists such a Y'. Then for all $i \in N(Y' \setminus Y)$, $Y'_i \setminus Y_i \subseteq Y'_i \subseteq \hat{C}_i(Y'_i|Y'_{-i})$, a contradiction since Y is stable (and therefore unblocked) given $\{\hat{C}_i\}_{i\in I}$. It follows from Theorem S.1 that Y is stable for some strategically consistent assessment $\{C_i, \mu_i\}_{i\in I}$ satisfying weak forward induction. □

S.4 Multilateral vs. Bilateral Contracts

We emphasize that interactions between pairs of agents specified by a multilateral contract cannot be represented as independent bilateral contracts in a matching on networks model. We illustrate this fact using an example from professional sports.

Example S.2 (Multilateral vs. Bilateral Agreements). On July 31, 2014, Major League Baseball's Detroit Tigers, Tampa Bay Rays, and Seattle Mariners traded a total of five players as part of a single multilateral agreement. Detroit sent two players to Tampa Bay; Tampa Bay sent one player to Detroit; Detroit sent one player to Seattle; and Seattle sent one player to Tampa Bay.[12]

If we tried to model this transaction as three independent bilateral contracts, instead of a single multilateral contract, we would generally fail to predict that it would take place. Indeed, a contract representing the bilateral interaction between the Seattle Mariners and Detroit Tigers would not be individually rational for the Tigers, as they would send Seattle a valuable player without receiving anything in return. Likewise, a contract representing the bilateral interaction between the Mariners and Rays would not be individually rational for the Mariners. These interactions are only possible because they were conducted as part of a single multilateral agreement.

S.5 Strategic Consistency and Nash Equilibrium

Consider the normal form game G defined as follows:

Players The agents $i \in I$.

Actions Sets of contracts $S_i \in 2^{X_i}$.

Payoffs $\pi_i(\{S_i\}_{i\in I}) = u_i\left(\cap_{i\in I}(S_i \cup X_{-i})\right)$.

G extends the link-announcement game discussed in, e.g., Myerson (1991) and Jackson (2010) to our matching with contracts setting. Proposition S.1 shows that an outcome is nonstrategically individually rational if and only if it is the set of contracts signed in a Nash equilibrium of G. The nonstrategically individually rational outcomes are precisely those that can be pinned down by the choice functions and beliefs in a strategically consistent profile

[12]Source: http://mlb.mlb.com/mlb/transactions/index.jsp#month=7&year=2014.

(Lemmas 1 and 2), so a profile of choice functions and beliefs is strategically consistent if and only if it is cross-set consistent and maps each set of contracts Y to the contracts signed in a pure strategy Nash equilibrium of G.

Proposition S.1 (Nonstrategic Individual Rationality and Nash Equilibrium). *If Y is nonstrategically individually rational, then $\{Y_i\}_{i\in I}$ is a Nash equilibrium of G. Conversely, if $\{S_i(Y)\}_{i\in I}$ is a Nash equilibrium of G, then $S(Y) \equiv \bigcap_{i\in I}(S_i(Y) \cup X_{-i})$ is nonstrategically individually rational.*

Proposition S.2 (Strategic Consistency and Nash Equilibrium). *If $\{C_i, \mu_i\}_{i\in I}$ is a strategically consistent profile of choice functions and beliefs, then for each $Y \subseteq X$, $\{C_i(Y_i|Y_{-i})\}_{i\in I}$ is a Nash equilibrium of G.*

Conversely, if for each $Y \subseteq X$, there is a Nash equilibrium $\{S_i(Y)\}_{i\in I}$ of G such that for each $i \in I$, $\mu_i(Y) = S(Y) \equiv \bigcap_{i\in I}(S_i(Y) \cup X_{-i})$ and $C_i(Y_i|Y_{-i}) = S(Y) \cap X_i$, then the choices $\{C_i\}_{i\in I}$ are optimal given $\{\mu_i\}_{i\in I}$, and the beliefs $\{\mu_i\}_{i\in I}$ are correct given $\{C_i\}_{i\in I}$. If, in addition, we have $S(Y) = S(Z)$ whenever $S(Y) \subseteq Z \subseteq Y$, then $\{C_i, \mu_i\}_{i\in I}$ is a strategically consistent profile.

S.5.1 Proofs for Appendix S.5

Proof of Proposition S.1 (Nonstrategic Individual Rationality and Nash Equilibrium) (Nash Equilibrium ⇒ Nonstrategic IR) Suppose that for each $Y \subseteq X$, $\{S_i(Y)\}_{i\in I}$ is a Nash equilibrium of G. By definition and since $S_i(Y) \subseteq X_i$, we have

$$S_i(Y) \cap \left(\bigcap_{j\neq i}(S_j(Y) \cup X_{-j})\right) = (S_i(Y) \cup X_{-i}) \cap \left(\bigcap_{j\neq i}(S_j(Y) \cup X_{-j})\right) \cap X_i = S(Y)_i, \text{ and}$$
$$\left(\bigcap_{j\neq i}(S_j(Y) \cup X_{-j})\right) \cap X_{-i} = (S_i(Y) \cup X_{-i}) \cap \left(\bigcap_{j\neq i}(S_j(Y) \cup X_{-j})\right) \cap X_{-i} = S(Y)_{-i}.$$

Then since $\{S_i(Y)\}_{i\in I}$ is a Nash equilibrium of G, for each $Y \subseteq X$ and $i \in I$,

$$S_i(Y) \in \arg\max_{S_i \subseteq X_i} u_i((S_i \cup X_{-i}) \bigcap_{j\neq i} (S_j(Y) \cup X_{-j}));$$

$$\Leftrightarrow S(Y)_i = S_i(Y) \cap \left(\bigcap_{j\neq i}(S_j(Y)\cup X_{-j})\right) \in \arg\max_{S_i \subseteq \left(\bigcap_{j\neq i}(S_j(Y)\cup X_{-j})\right)_i} u_i\left(S_i \cup \left(\bigcap_{j\neq i}(S_j(Y)\cup X_{-j})\right) \cap X_{-i}\right);$$

$$\Rightarrow S(Y)_i \in \arg\max_{S_i \subseteq S(Y)_i} u_i\left(S_i \cup \left(\bigcap_{j\neq i}(S_j(Y)\cup X_{-j})\right)\cap X_{-i}\right)$$

$$= \arg\max_{S_i \subseteq S(Y)_i} u_i\left(S_i \cup S(Y)_{-i}\right);$$

$$\Leftrightarrow S(Y)_i = \hat{C}_i(S(Y)_i | S(Y)_{-i}),$$

and so $S(Y)$ is nonstrategically individually rational.

(If) Suppose that Y is nonstrategically individually rational. Since $N(x) \geq 2$ for each $x \in X$, $\bigcap_{j\neq i}(Y_j \cup X_{-j}) = Y$. Then for each $i \in I$,

$$Y_i \in \arg\max_{S\subseteq Y_i} u_i(S \cup Y_{-i}) = \arg\max_{S_i \in 2^{Y_i}} u_i\left((S_i \cup X_{-i}) \cap \left(\bigcap_{j\neq i}(Y_j \cup X_{-j})\right)\right)$$

$$\subseteq \arg\max_{S_i \in 2^{X_i}} u_i\left((S_i \cup X_{-i}) \cap \left(\bigcap_{j\neq i}(Y_j \cup X_{-j})\right)\right).$$

It follows that $\{Y_i\}_{i\in I}$ is a Nash equilibrium of G. □

Proof of Proposition S.2 (Strategic Consistency and Nash Equilibrium) (Strategic Consistency ⇒ Nash Equilibrium) Suppose that $\{C_i, \mu_i\}_{i\in I}$ is strategically consistent. By Lemmas 1 and 2, at each set $Y \subseteq X$, $\mu_i(Y)$ is nonstrategically individually rational, and $C_i(Y_i|Y_{-i}) = \mu_i(Y) \cap X_i$, for each $i \in I$. Then from Proposition S.1, $\{C_i(Y_i|Y_{-i})\}_{i\in I}$ is a Nash equilibrium of G.

(Nash Equilibrium ⇒ Strategic Consistency) By Proposition S.1, $S(Y)$ is nonstrategically individually rational for each $Y \subseteq X$. Then by Lemma 2, $\{C_i\}_{i\in I}$ are optimal given $\{\mu_i\}_{i\in I}$ and $\{\mu_i\}_{i\in I}$ are correct given $\{C_i\}_{i\in I}$. If, in addition, $S(Y) = S(Z)$ whenever $S(Y) \subseteq Z \subseteq Y$, then for each $i \in I$, $\mu_i(Y) \subseteq Z \subseteq Y$ implies $\mu_i(Y) = \mu_i(Z)$; cross-set (and hence strategic) consistency of $\{C_i, \mu_i\}_{i\in I}$ then follows from Lemma 4. □

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