Strategic Consistency in Two-Sided Matching Markets*

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It is well known that in matching markets with externalities, complex preferences, or multilateral contracts, existence of stable outcomes is not always guaranteed (e.g., Sasaki and Toda (1996); Hatfield and Kojima (2008); Bando and Hirai (2021)). In Rostek and Yoder (2023), we show that when the agents behave in a *strategically consistent* manner — that is, form correct beliefs about each other's choices from each set of contracts that might be available, choose optimally given those beliefs, and form beliefs in a way that is consistent across different sets of available contracts — stable outcomes always exist, even when the market has these features.

In this note, we focus on *two-sided* markets — such as those between doctors and hospitals, or firms and consumers — where externalities are present, such as those studied by Pycia and Yenmez (2023). This allows us to consider a weaker form of strategic consistency, *within-side strategic consistency*, where agents only form beliefs about the behavior of other agents on the same side of the market. This requires less strategic sophistication than strategic consistency does in more general settings. In particular, an agent does not need to make inference about the contracts that others will choose to sign *with him*. Instead, he only needs to infer the contracts that others on his side of the market will choose to sign with agents on the *opposite side of the market*.

We give two main sets of results. First, we show that together with substitutability, within-side strategic consistency ensures that stable outcomes exist. Moreover, like in classi-

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cal two-sided matching markets, the set of all such outcomes has a lattice structure. However, unlike with the full strategic consistency condition considered in Rostek and Yoder (2023), stable outcomes are not always uniquely pinned down by a *within-side* strategically consistent profile of choice functions and beliefs.

Second, we give conditions on the model's primitives that ensure that there are profiles of choice functions and beliefs that satisfy within-side strategic consistency and substitutability. These sufficient conditions are precisely those introduced to this setting by Pycia and Yenmez (2023) to ensure that stable outcomes exist when agents do not make inferences about the choices of others. This clarifies the connection between our results from Rostek and Yoder (2023) and theirs. In particular, it shows that in two-sided markets with externalities, the conditions they introduce ensure that agents can form correct beliefs about the behavior of others on the same side of the market that are consistent across sets of available contracts; hence, they ensure existence even in the presence of strategic sophistication *within* each side of the market.

1 Model

We consider the same setting as Rostek and Yoder (2023), specialized to a two-sided market with externalities.

There is a finite set of agents I that can be partitioned into two *sides* $J, K \subseteq I$,[1] and a finite set of agreements X they can make with each other. Contracting is bilateral, and occurs only with agents on the other side of the market: Each agreement x requires the agreement of exactly one agent $j_x \in J$ and one agent $k_x \in K$ to go into effect. For each agent $i \in I$, label the set of contracts that require i's agreement as $X_i \equiv \{x \mid i \in \{j_x, k_x\}\}$. Similarly, we label $X_{-i} \equiv X \setminus X_i$, and for sets of contracts $Y \subseteq X$, we write $Y_i \equiv Y \cap X_i$ and $Y_{-i} \equiv Y \cap X_{-i}$.

Each agent i has preferences over sets of contracts that take effect, or *outcomes*, which are represented by a utility function $u_i : 2^X \to \mathbb{R}$. This accommodates contracts that have *externalities*: When j_x and k_x make an agreement x, it can affect not just their utility, but the utility of other agents as well. In the absence of such contracts — i.e., when $u_i(Y \cup Z) = u_i(Y \cup Z')$ for each $Z, Z' \subseteq X_{-i}$ and $i \in I$ — we say that there are *no externalities*.

[1]That is, $J \cup K = I$ and $J \cap K = \emptyset$.

A *choice function* for agent i is a function $C_i : 2^{X_i} \times 2^{X_{-i}} \to 2^{X_i}$. When agent i's choice function is C_i, $C_i(Y_i|Y_{-i})$ gives the set of contracts that agent i chooses from the set of available contracts Y_i, given the contracts in Y_{-i}. Its second argument allows for the presence of externalities. In Section 2, we describe two different ways in which we can construct these choice functions from agents' preferences. To guarantee that both will yield single-valued choice functions, we assume that agents are never indifferent about sets of contracts that they might agree to, holding fixed the set of contracts that they are not involved in: $u_i(Y \cup X') \neq u_i(Z \cup X')$ for each distinct $Y, Z \subseteq X_i$ and $X' \subseteq X_{-i}$.

A key condition on choice functions — both in the literature, and in this paper — is *substitutability*. We say that C_i is *substitutable* if the *rejection function* $R_i(Y_i|Y_{-i}) \equiv Y_i \setminus C_i(Y_i|Y_{-i})$ is monotone in both arguments.[2]

1.1 Stability

We use the standard matching-theoretic solution concept, *stability*, generalized to allow for externalities.

Definition (Stability). Given choice functions $\{C_i\}_{i \in I}$, a set of contracts $Y \subseteq X$ is *stable* if it is

i. *Individually rational*: $Y_i = C_i(Y_i|Y_{-i})$ for all $i \in N$.

ii. *Unblocked*: There does not exist $Z \subseteq (X \setminus Y)$ such that for all $i \in N(Z)$, $Z_i \subseteq C_i((Z \cup Y)_i|(Z \cup Y)_{-i})$.

In words, a set of contracts Y is stable if (i) when Y is the set of available contracts, no one rejects any contracts from it (individual rationality), and (ii) no group of agents can propose a new set of contracts Z, or *block*, that they are each willing to choose when made available alongside Y.

We accommodate externalities by allowing agents who participate in a block to take into account the contracts available to the agents they negotiate with: the second argument of

[2] This extends the substitutes condition to encompass externalities in a different way than Pycia and Yenmez (2023): Ours implies that contracts one agent might sign are substitutable for contracts involving other agents, while theirs implies that one agent's contracts are substitutable for *better outcomes* for other agents.

the choice function in (ii) includes both the existing contracts Y_{-i} and blocking contracts Z_{-i} that do not name agent i.[3]

2 Within-Side Strategic Consistency

We begin by modifying the definition of strategic consistency from Rostek and Yoder (2023) so that agents only form beliefs about the behavior of other agents on the same side of the market.

Definition (Within-Side Strategic Consistency and Nonstrategic Choice). Given agents' payoffs $\{u_i : 2^X \to \mathbb{R}\}_{i\in I}$,

- A profile of choice functions $\{C_i : 2^{X_i} \times 2^{X_{-i}} \to 2^{X_i}\}_{i\in I}$ and *beliefs* $\{\mu_i : 2^X \to 2^{X_{-i}}\}_{i\in I}$ is *within-side strategically consistent* if for each side $L \in \{J, K\}$ and each $i \in L$,

  i. μ_i is *correct* given $\{C_j\}_{\substack{j\in L\\ j\neq i}}$: For each $Y \subseteq X$, $\mu_i(Y) = C_{L-i}(Y) \equiv \bigcup_{\substack{j\in L\\ j\neq i}} C_j(Y_j|Y_{-j})$.

  ii. C_i is *optimal* given μ_i: For each $Y \subseteq X$, $C_i(Y_i|Y_{-i}) = \arg\max_{S\subseteq Y_i} u_i(S \cup \mu_i(Y))$.

  iii. μ_i is *cross-set consistent* given $\{C_i\}_{i\in L}$: For each $Y, Z \subseteq X$, if $Y \supseteq Z \supseteq C_j(Y_j|Y_{-j})$ for all $j \in L$, then $\mu_i(Z) = \mu_i(Y)$.

- Each agent i's *nonstrategic choice function* $\hat{C}_i$ is defined by $\hat{C}_i(Y_i|Y_{-i}) = \arg\max_{S\subseteq Y_i} u_i(S\cup Y_{-i})$.

Two epistemic assumptions motivate within-side strategic consistency. First, when faced with a set of contracts that has been proposed to their side of the market, they form correct beliefs about which contracts the other agents on the same side will choose. Second, their beliefs do not change when contracts that are not chosen by any agent on the same side are removed. Nonstrategic choice functions, on the other hand, describe how agents will choose when they assume that all of the contracts available to others on the same side of the market will go into effect.

[3] We generalize the usual definition of stability to accommodate externalities in a slightly different way than Pycia and Yenmez (2023). Under the definition they adopt, agents in a blocking coalition do not anticipate any changes to the set of contracts signed by other agents, even the other members of the blocking coalition. Our stability definition instead assumes that agents in a blocking coalition account for the contracts added by the agents they negotiate with.

Because the correctness and optimality conditions define a fixed point condition, we can think of the choice functions in a within-side strategically consistent profile as being an equilibrium object. This contrasts with the standard approach to choice in the matching literature (nonstrategic choice), which pins down agents' choice functions as the solution to a single-agent optimization problem.

In classical two-sided models with no externalities, agents' beliefs about the behavior of others on the same side do not affect their choices, since they are not involved in or affected by the contracts those other agents sign. Hence, in such settings, nonstrategic choice pins down agents' behavior in the same way as within-side strategic consistency.[4] But more generally — e.g., with externalities, multilateral contracts, or a market structure without two sides — nonstrategic and strategically consistent choice need not coincide.

2.1 Stable Outcomes

In this section, we show that with within-side strategic consistency, the standard characterization of stable outcomes in two-sided markets extends to environments with externalities. In particular, Theorem 1 shows that with externalities, the Gale-Shapley algorithm functions (i.e., finds every stable outcome) when agents' choice functions satisfy substitutability (as is standard) and are part of a within-side strategically consistent profile of choice functions and beliefs. To do so, we first show that these conditions allow us to characterize stability in terms of a system of equations, as in Hatfield and Milgrom (2005). This system of equations features the two sides' *aggregate rejection functions* $R_L : 2^X \to 2^X$ defined by $R_L(Y) \equiv Y \setminus \left(\bigcup_{i \in L} C_i(Y_i | Y_{-i})\right)$.

Lemma 1 (Stability as a Fixed Point). *Suppose that $\{C_i, \mu_i\}_{i \in I}$ is a within-side strategically consistent profile of choice functions and beliefs, and that the choice functions $\{C_i\}_{i \in I}$ satisfy substitutability. Then X' is stable for $\{C_i\}_{i \in I}$ if and only if $X' = Y \cap Z$ for some solution (Y, Z) to the system of equations*

$$Y = X \setminus R_J(Z), \qquad Z = X \setminus R_K(Y). \tag{1}$$

Lemma 1 allows us to define a monotone operator F whose fixed points correspond to

[4]That is, in a two-sided market without externalities, choice functions are part of a within-side strategically consistent profile if, and only if, they are nonstrategic.

stable outcomes: Define $F : 2^X \times 2^X \to 2^X \times 2^X$ by

$$F_1(Y,Z) \equiv X \setminus R_J(Z), \qquad\qquad F_2(Y,Z) \equiv X \setminus R_K(F_1(Y,Z)).$$

When the agents' choice functions satisfy substitutability, F is monotone in the partial order $\succeq_F \equiv (\supseteq, \subseteq)$. Lemma 1 implies that when they are also part of a within-side strategically consistent profile, the set $\{Y \cap Z \mid F(Y,Z) = Y, Z\}$ is exactly the set of stable outcomes.

As in Hatfield and Milgrom (2005), iterated applications of F correspond to rounds of a generalized Gale-Shapley algorithm in which agents in one group make offers and agents in the other conditionally accept them. Tarski's theorem ensures that this algorithm converges to a fixed point of F, and thus — so long as agents' behavior in the algorithm is within-side strategically consistent — a stable outcome. It also ensures that these fixed points form a lattice.[5]

Theorem 1 (Within-Side Strategic Consistency and Stability). *Suppose that $\{C_i, \mu_i\}_{i \in I}$ is a within-side strategically consistent profile of choice functions and beliefs, and that the choice functions $\{C_i\}_{i \in I}$ satisfy substitutability. Then the set of outcomes stable for $\{C_i\}_{i \in I}$ is given by $\{Y \cap Z \mid F(Y,Z) = (Y,Z)\}$, and the set of fixed points of F is a nonempty lattice with the order $\succeq_F \equiv (\supseteq, \subseteq)$.*

In Theorem 1, within-side strategic consistency ensures that contracts rejected over the course of the Gale-Shapley algorithm are unable to block the algorithm's outcome. Without it, for instance, a pair of agents j_x, j_y on the same side J might each reject contracts x and y that are offered to them simultaneously by the algorithm, under the incorrect assumption that the other agent will choose the contract that he was offered. But if, e.g., j_x would choose x when it is offered alone, x could still be part of a block once the algorithm terminates.

2.2 Within-Side Strategic Consistency: Existence

In a two-sided market with externalities, it may not be possible for agents on the same side of the market to simultaneously make correct predictions about each other's behavior. That

[5]In particular, it can also be shown that (as is standard) starting from $(\emptyset, X)$, the generalized Gale-Shapley algorithm converges to the $\succeq_F$-smallest fixed point $(\underline{Y}, \underline{Z})$ of F, and starting from $(X, \emptyset)$, it converges to F's $\succeq_F$-largest fixed point $(\overline{Y}, \overline{Z})$.

is, a strategically consistent assessment for each side of the market may not exist. Here, we give conditions on preferences — in the form of conditions on the nonstrategic choice functions they give rise to — which are sufficient for the existence of strategically consistent assessments for each side of the market such that choice functions satisfy substitutability. These conditions are familiar from Pycia and Yenmez (2023); in order to present our result, we restate them here.

$\hat{C}_i$ satisfies *standard substitutability* if its rejection function $\hat{R}_i$ is monotone in its first argument: For all $Y, Z \subseteq X_i$ and $X' \subseteq X_{-i}$, $Y \subseteq Z \Rightarrow \hat{R}_i(Y|X') \subseteq \hat{R}_i(Z|X')$.

$\hat{C}_i$ satisfies *irrelevance of rejected contracts* if, whenever an agent with choice function $\hat{C}_i$ rejects contracts, making those contracts unavailable does not change his choices: For all $Y, Z \subseteq X_i$ and $X' \subseteq X_{-i}$, $\hat{C}_i(Z|X') \subseteq Y \subseteq Z \Rightarrow \hat{C}_i(Y|X') = \hat{C}_i(Z|X')$.

For a side of the market $L \in \{J, K\}$, the preorder $\succeq_L$ on 2^X is *consistent* with $\{\hat{C}_i\}_{i\in L}$ if $\bigcup_{i\in J} \hat{C}_i(Y'_i|Z'_{-i}) \succeq \bigcup_{i\in L} \hat{C}_i(Y_i|Z_{-i})$ for each $Y' \supseteq Y$ and $Z' \succeq Z$. $\{\hat{C}_i\}_{i\in I}$ satisfy *monotone externalities* if for each $L \in \{J, K\}$, there is a preorder $\succeq_L$ consistent with $\{\hat{C}_i\}_{i\in L}$ such that for all $Y \subseteq X$ and $Z' \succeq Z$, $\hat{R}_i(Y_i|Z'_{-i}) \supseteq \hat{R}_i(Y_i|Z_{-i})$ for each $i \in L$.

Theorem 2 (Monotone Externalities as a Foundation For Strategic Consistency). *If agents' nonstrategic choice functions $\{\hat{C}_i\}_{i\in I}$ satisfy irrelevance of rejected contracts, standard substitutability, and monotone externalities, then there is a within-side strategically consistent profile $\{C_i, \mu_i\}_{i\in I}$ of choice functions and beliefs such that each C_i satisfies subsitutability.*

Theorem 2 shows that in a two-sided matching setting where agents' nonstrategic choice functions satisfy Pycia and Yenmez' (2023) conditions, it is always possible for agents on the same side of the market to simultaneously make correct predictions about each other's behavior, even if they are not strategic about the behavior of agents on the opposite side of the market. Hence, Theorem 1 implies that when these conditions are satisfied and agents are strategically sophisticated about the behavior of others on the *same* side of the market, stable outcomes exist. This complements the main result of Pycia and Yenmez (2023), who show that these conditions ensure the existence of stable outcomes when agents behave nonstrategically.

Note, however, that unlike in Rostek and Yoder (2023) (Theorem 2), conditions on preferences are necessary for existence in both Theorem 2 and in the setting of Pycia and Yenmez

(2023) because agents are not strategically sophisticated about the behavior of *every* other agent, and so make implicit assumptions about others' choices that may be inaccurate. In both settings, substitutability conditions ensure that these inaccuracies are not relevant for stability: In particular, even when agents make the (possibly incorrect) assumption that agents on the other side of the market will not drop any existing contracts as part of a block, substitutability ensures that they never choose blocking contracts that they would reject if their beliefs were correct. In Theorem 2, however, these conditions play the additional role of ensuring that inaccurate assumptions about the behavior of agents on the *other* side of the market do not interfere with the ability of agents on the *same* side to have a consistent system of choices and beliefs.

Interestingly, Theorem 1 also implies that under these conditions, when the agents' correct beliefs about others' behavior — and thus the strategically consistent profile — are held fixed, the set of stable outcomes forms a lattice.[6] This contrasts with settings where agents behave nonstrategically: there, as Pycia and Yenmez (2023) show, the set of stable outcomes does not have a lattice structure.

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[6]However, we should not expect that the set of outcomes that are stable for *some* strategically consistent assessment will have a lattice characterization: If 2^X is not a lattice in the consistent preorder $\succeq_J$, then the set of fixed points of $G_Y(Z) \equiv \bigcup_{i \in J} \hat{C}_i(Y_i|Z_{-i})$ — and hence the set of choice function profiles generated by the externalities in nonstrategic choice — does not form a lattice.

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Appendix

To begin, for each side $L \in \{J, K\}$, define its *aggregate choice function* $C_L : 2^X \to 2^X$ as $C_L(Y) \equiv \cup_{i \in L} C_i(Y_i|Y_{-i})$.

Lemma 2. *Let $L \in \{J, K\}$, and suppose that beliefs $\{\mu_i\}_{i \in L}$ are correct given choice functions $\{C_i\}_{i \in L}$. Then $\{\mu_i\}_{i \in L}$ are cross-set consistent given $\{C_i\}_{i \in L}$ if and only if $Y \supseteq Z \supseteq C_L(Y)$ implies $C_L(Y) = C_L(Z)$.*

Proof. By definition, for each $i, j \in L$, X_i and X_j are disjoint. Then since $\{\mu_i\}_{i \in L}$ are correct given $\{C_i\}_{i \in L}$, $C_L(Y) = C_L(Z) \Leftrightarrow C_{L-i}(Y) = C_{L-i}(Z)$ for each $i \in L \Leftrightarrow \mu_i(Y) = \mu_i(Z)$ for each $i \in L$. Moreover, by definition, $Y \supseteq Z \supseteq C_L(Y) \Leftrightarrow Y \supseteq Z \supseteq C_j(Y_j|Y_{-j})$ for each $j \in I$. The claim then follows from the definition of cross-set consistency. □

Lemma 3 (Stability in Aggregate). *Y is stable for $\{C_i\}_{i \in I}$ if and only if* 1. *$C_J(Y) = C_K(Y) = Y$ and* 2. *$Y' \nsubseteq C_L(Y' \cup Y) \cap C_K(Y' \cup Y)$ for all $Y' \nsubseteq Y$.*

Proof. ($\Rightarrow$) Suppose Y is stable. We begin by proving condition 1: Since Y is individually rational, $Y_i = C_i(Y_i|Y_{-i})$ for all $i \in I$. Then from the definition of C_L, $Y = C_K(Y) = C_J(Y)$.

Now suppose that condition 2 fails, and there exists $Y' \nsubseteq Y$ such that $Y' \subseteq C_J(Y' \cup Y) \cap C_K(Y' \cup Y)$. From the definition of C_L, for each $L \in \{J, K\}$ and $i \in L$, $Y' \subseteq C_L(Y' \cup Y) \subseteq C_i((Y' \cup Y)_i|(Y' \cup Y)_{-i}) \cup (Y' \cup Y)_{-i}$, and hence $Y'_i \subseteq C_i((Y' \cup Y)_i|(Y' \cup Y)_{-i})$. Then Y' blocks Y, a contradiction.

($\Leftarrow$) Suppose that conditions 1 and 2 hold. Y is individually rational: By definition of C_L, for each $L \in \{J, K\}$ and $i \in L$, condition 1 implies $Y_i = C_L(Y) \cap X_i = C_i(Y_i|Y_{-i})$. .

Y is unblocked: Suppose not, and there exists $Z \subseteq X \setminus Y$ such that $Z_i \subseteq C_i((Z \cup Y)_i|(Z \cup Y)_{-i})$ for each $i \in I$. Then for each $L \in \{J, K\}$, by definition of C_L, and since $\bigcup_{i \in L} X_i = X$, $Z \subseteq C_L(Y \cup Z)$. Then $Z \subseteq C_J(Y \cup Z) \cap C_K(Y \cup Z)$, contradicting condition 2. □

Lemma 4. *For $L \in \{J, K\}$, if $\{C_i\}_{i \in L}$ are substitutable, then R_L is monotone.*

Proof. By definition, for each $i, j \in L$, $X_i \cap X_j = \emptyset$. Then we have $R_L(Y) \equiv Y \setminus \left(\bigcup_{i \in L} C_i(Y_i|Y_{-i})\right) = \bigcap_{i \in L}(Y \setminus C_i(Y_i|Y_{-i})) = \bigcup_{i \in L} R_i(Y)$. The claim follows. □

Proof of Lemma 1 (Stability as a Fixed Point) (((Y,Z) solves (1)$\Rightarrow Y \cap Z$ is stable): First we show that $Y \cap Z$ satisfies Lemma 3's condition 1. Suppose (Y,Z) solves (1). Then $Y \cap Z = Z \setminus R_J(Z) = C_J(Z)$. Similarly, $Y \cap Z = Y \setminus R_K(Y) = C_K(Y)$. Since $\{C_i, \mu_i\}_{i\in I}$ is within-side strategically consistent, by Lemma 2, $C_K(Y) = C_K(Y \cap Z) = Y \cap Z = C_J(Z) = C_J(Y \cap Z)$.

Now we show that $Y \cap Z$ satisfies Lemma 3's condition 2. By Lemma 2, for any $Y' \not\subseteq Y \cap Z$, $C_J((Y' \cap Z) \cup (Y \cap Z)) = Y \cap Z$.

Since $\{C_i\}_{i\in I}$ are substitutable, by Lemma 4, R_L and R_K are monotone. Then $R_J(Y' \cup (Y \cap Z)) \supseteq R_J((Y' \cap Z) \cup (Y \cap Z)) = (Y' \setminus Y) \cap Z$. Then set arithmetic yields

$$
\begin{aligned}
C_J(Y' \cup (Y \cap Z)) &= (Y' \cup (Y \cap Z)) \setminus R_J(Y' \cup (Y \cap Z)),\\
&\subseteq (Y' \cup (Y \cap Z)) \setminus ((Y' \setminus Y) \cap Z),\\
&= ((Y' \setminus Z) \cup ((Y' \cup Y) \cap Z)) \setminus ((Y' \setminus Y) \cap Z),\\
&= (Y' \setminus Z) \cup (Y \cap Z) = (Y' \setminus Z) \cup C_J(Z),\\
&\subseteq (X \setminus Z) \cup C_J(Z) = X \setminus R_J(Z) = Y.
\end{aligned}
$$

Likewise, $C_K(Y' \cup (Y \cap Z)) \subseteq Z$, so $C_J(Y' \cup (Y \cap Z)) \cap C_K(Y' \cup (Y \cap Z)) \subseteq Y \cap Z$. It follows that $Y \cap Z$ satisfies Lemma 3's condition 2. Thus, by Lemma 3, $Y \cap Z$ is stable.

(X' is stable $\Rightarrow X' = Y \cap Z$ for some (Y,Z) satisfying (1)): Suppose X' is stable, and let

$$
Y = \bigcup_{x\in X} C_J(X' \cup \{x\}), \qquad Z = (X \setminus Y) \cup X'.
$$

First, we show that $X' = Y \cap Z$: By Lemma 3, $X' = C_K(X') = C_J(X')$. Then for any $x \in X'$, $C_J(X' \cup \{x\}) = C_J(X') = X'$. Then $X' \subseteq Y$, implying $Y \cap Z = X'$.

Next, we show that (Y,Z) solve (1). We start by showing $Z = X \setminus R_K(Y)$: By construction, for any $x \in Y \setminus X'$, we must have $x \in C_J(X' \cup \{x\})$. Since X' is stable, it must satisfy Lemma 3's condition 2. Then we must have $x \notin C_K(X' \cup \{x\})$, or equivalently, $x \in R_K(X' \cup \{x\})$, for any $x \in Y \setminus X'$. Then since $\{C_i\}_{i\in I}$ are substitutable, by Lemma 4, $x \in R_K(Y)$ for any $x \in Y \setminus X'$; equivalently, $Y \setminus X' \subseteq R_K(Y)$. Then by definition of R_K and C_K, we have $C_K(Y) \subseteq X'$. By Lemma 3, since X' is stable, $C_K(X') = X'$. Then since $\{C_i, \mu_i\}_{i\in I}$ is within-side strategically consistent, and since $X' = Y \cap Z \subseteq Y$, by Lemma 2, $C_K(Y) = X'$. So $Z = X \setminus R_K(Y)$.

We conclude by showing that $Y = X \setminus R_K(Z)$. We have shown that $X' = Y \cap Z \subseteq Y$. Then for all $x \in Z \setminus X'$, $x \notin Y$, and so $x \notin C_J(X' \cup \{x\})$, or equivalently, $x \in R_J(X' \cup \{x\})$. Then since $\{C_i\}_{i \in I}$ are substitutable, by Lemma 4, for all $x \in Z \setminus X'$, $x \in R_J(Z)$; equivalently, $Z \setminus X' \subseteq R_J(Z)$. Then by construction of C_J and R_J, $C_J(Z) \subseteq X'$.

Since X' is stable, by Lemma 3, $C_J(X') = X'$. Then since $\{C_i, \mu_i\}_{i \in I}$ is within-side strategically consistent, by Lemma 2, $C_J(Z) = X'$. So $Y = X \setminus R_J(Z)$, as desired. □

Proof of Theorem 1 (Within-Side Strategic Consistency and Stability) The set of stable outcomes is $\{Y \cap Z | F(Y, Z) = (Y, Z)\}$: If $F(Y, Z) = (Y, Z)$, then $Y = F_1(Y, Z) = X \setminus R_J(Z)$ and $Z = F_2(Y, Z) = X \setminus R_K(F_1(Y, Z)) = X \setminus R_K(Y)$, so by Lemma 1, $Y \cap Z$ is stable. Conversely, if X' is stable, then by Lemma 1, there exist $Y, Z \subseteq X$ such that $X' = Y \cap Z$, $Y = X \setminus R_J(Z)$, and $Z = X \setminus R_K(Y)$. Then $Y = F_1(Y, Z)$; consequently, $Z = X \setminus R_K(F_1(Y, Z)) = F_2(Y, Z)$ as well.

F's fixed points are a nonempty lattice in $\succeq_F$: Since $\{C_i\}_{i \in I}$ are substitutable, by Lemma 4, R_J and R_K are monotone. Then F is monotone in $\succeq_F$, and so by Tarski's fixed point theorem, the fixed points of F form a nonempty lattice with the order $\succeq_F$. □

Proof of Theorem 2 (Monotone Externalities as a Foundation For Strategic Consistency) Suppose that agents' nonstrategic choice functions $\{\hat{C}_i\}_{i \in I}$ satisfy irrelevance of rejected contracts, standard substitutability, and monotone externalities, and let $L \in \{J, K\}$. We construct choice functions and beliefs $\{C_i, \mu_i\}_{i \in L}$ for the agents in L in terms of the minimal fixed points of the functions $G_Y(Z) \equiv \bigcup_{i \in L} \hat{C}_i(Y_i | Z_{-i})$, which we show exist for each Y in Claim 1. Claims 2-5 show that these fixed points have properties that ensure that these beliefs are each correct and cross-set consistent given the profile of choice functions (Claims 6 and 8) and these choice functions are each optimal given the profile of beliefs and substitutable (Claims 7 and 9). Hence, the profile $\{C_i, \mu_i\}_{i \in I}$ of *both sides'* choice functions and beliefs is within-side strategically consistent.

Claim 1. For each $Y \subseteq X$, $G_Y(Z) \equiv \bigcup_{i \in L} \hat{C}_i(Y_i | Z_{-i})$ has a $\succeq_J$-minimal fixed point Y^*. For any $Y \subseteq X$, define the sequence $\{\hat{Y}^t\}$ recursively as follows:

$$\hat{Y}^0 = \emptyset, \qquad \hat{Y}^{t+1} = \bigcup_{i \in L} \hat{C}_i(Y_i | \hat{Y}^t_{-i})$$

Let $\succeq_L$ be the consistent preorder for which $\{\hat{C}_i\}_{i\in L}$ satisfy monotone externalities. We show that $\{\hat{Y}^t\}$ is $\succeq$-increasing. For $t=0$, since $\succeq_L$ is consistent with $\{\hat{C}_i\}_{i\in L}$, we have

$$\hat{Y}^1 = \bigcup_{i\in L} \hat{C}_i(Y_i|\emptyset) \succeq_L \bigcup_{i\in L} \hat{C}_i(\emptyset|\emptyset) = \emptyset = \hat{Y}^0.$$

Now for $t \geq 1$, suppose $\hat{Y}^t \succeq_L \hat{Y}^{t-1}$. Then since $\succeq_L$ is consistent with $\{\hat{C}_i\}_{i\in L}$,

$$\hat{Y}^{t+1} = \bigcup_{i\in L} \hat{C}_i(Y_i|\hat{Y}^t_{-i}) \succeq_L \bigcup_{i\in L} \hat{C}_i(Y_i|\hat{Y}^{t-1}_{-i}) = \hat{Y}^t.$$

It follows by induction that $\hat{Y}^{t+1} \succeq_L \hat{Y}^t$ for each n. Let $T = |2^X| = 2^{|X|}$. Since $\succeq_L$ is a preorder, it is transitive. Then we must have $\hat{Y}^T \simeq \hat{Y}^t$ for each $t > T$. By monotone externalities, we have $R_i(Y_i|\hat{Y}^T_{-i}) \subseteq R_i(Y_i|\hat{Y}^{T+1}_{-i})$ and $R_i(Y_i|\hat{Y}^T_{-i}) \supseteq R_i(Y_i|\hat{Y}^{T+1}_{-i})$ for each $i \in L$. Then $\hat{C}_i(Y_i|\hat{Y}^T_{-i}) = \hat{C}_i(Y_i|\hat{Y}^{T+1}_{-i})$ for each $i \in L$. Then $\hat{Y}^{T+1} = \bigcup_{i\in L} \hat{C}_i(Y_i|\hat{Y}^T_{-i}) = \bigcup_{i\in L} \hat{C}_i(Y_i|\hat{Y}^{T+1}_{-i}) = \hat{Y}^{T+2}$. Thus $Y^* \equiv \hat{Y}^{T+1}$ is a fixed point of G_Y.

Now suppose there is some other fixed point $Y' \neq Y^*$ of G_Y such that $Y' \preceq_L Y^*$. Since $\succeq_L$ is consistent with $\{\hat{C}_i\}_{i\in L}$,

$$Y' = \bigcup_{i\in L} \hat{C}_i(Y_i|Y'_{-i}) \succeq_L \bigcup_{i\in L} \hat{C}_i(\emptyset|Y'_{-i}) = \emptyset = \hat{Y}^0.$$

Now for $t \geq 1$, suppose $Y' \succeq_L \hat{Y}^{t-1}$. Then since $\succeq_L$ is consistent with $\{\hat{C}_i\}_{i\in L}$,

$$Y' = \bigcup_{i\in L} \hat{C}_i(Y_i|Y'_{-i}) \succeq_L \bigcup_{i\in L} \hat{C}_i(Y_i|\hat{Y}^{t-1}_{-i}) = \hat{Y}^t.$$

It follows by induction that $Y' \succeq_L \hat{Y}^{T+1} = Y^*$. Then since $\{\hat{C}_i\}_{i\in L}$ satisfy monotone externalities, we have $R_i(Y_i|Y'_{-i}) \subseteq R_i(Y_i|Y^*_{-i})$ and $R_i(Y_i|Y'_{-i}) \subseteq R_i(Y_i|Y^*_{-i})$ for each $i \in L$. Then $\hat{C}_i(Y_i|Y'_{-i}) = \hat{C}_i(Y_i|Y^*_{-i})$ for each $i \in L$. Then $Y' = \bigcup_{i\in L} \hat{C}_i(Y_i|Y'_{-i}) = \bigcup_{i\in L} \hat{C}_i(Y_i|Y^*_{-i}) = Y^*$, a contradiction.

Claim 2. $Y^* \preceq_L Z^*$ **for each** $Y \subseteq Z$**.** By definition, $\hat{Z}^0 = \hat{Y}^0 = \emptyset$. Now for $t \geq 1$, suppose $\hat{Z}^{t-1} \succeq_L \hat{Y}^{t-1}$. Since $\succeq_L$ is consistent with $\{\hat{C}_i\}_{i\in L}$,

$$\hat{Z}^t = \bigcup_{i\in L} \hat{C}_i(Z_i|\hat{Z}^{t-1}_{-i}) \succeq_L \bigcup_{i\in L} \hat{C}_i(Y_i|\hat{Y}^{t-1}_{-i}) = \hat{Y}^t.$$

It follows by induction that $Y^* = \hat{Y}^{T+1} \preceq_L \hat{Z}^{T+1} = Z^*$.

Claim 3. If $Z^* \subseteq Y \subseteq Z$, then $Y^* \supseteq Z^*$. By Claim 2, we have $Y^* \preceq_L Z^*$. Since $\{\hat{C}_i\}_{i\in L}$ satisfy monotone externalities, for each $i \in L$, $\hat{R}_i(Y_i|Y^*_{-i}) \subseteq \hat{R}_i(Y_i|Z^*_{-i})$, or equivalently, $\hat{C}_i(Y_i|Y^*_{-i}) \supseteq \hat{C}_i(Y_i|Z^*_{-i})$. And since $\{\hat{C}_i\}_{i\in L}$ satisfy irrelevance of rejected contracts, $\hat{C}_i(Y_i|Z^*_{-i}) = \hat{C}_i(Z_i|Z^*_{-i})$ for each $i \in L$. Then

$$Y^* = \bigcup_{i\in L} \hat{C}_i(Y_i|Y^*_{-i}) \supseteq \bigcup_{i\in L} \hat{C}_i(Y_i|Z^*_{-i}) = \bigcup_{i\in L} \hat{C}_i(Z_i|Z^*_{-i}) = Z^*.$$

Claim 4. If $Z^* \subseteq Y \subseteq Z$, then $Y^* = Z^*$. By definition, we have $(Z^*)^* = \bigcup_{i\in L} \hat{C}_i(Z^*_i|(Z^*)^*_{-i}) \subseteq Z^*$. And by Claim 3, we have $(Z^*)^* \supseteq Z^*$. Then $(Z^*)^* = Z^*$.

Now by Claim 2, we have $Z^* = (Z^*)^* \preceq_L Y^* \preceq_L Z^*$. Then since $\{\hat{C}_i\}_{i\in L}$ satisfy monotone externalities, for each $i \in L$, $\hat{R}_i(Y_i|Y^*_{-i}) \subseteq \hat{R}_i(Y_i|Z^*_{-i}) \subseteq \hat{R}_i(Y_i|Y^*_{-i})$, or equivalently, $\hat{C}_i(Y_i|Y^*_{-i}) = \hat{C}_i(Y_i|Z^*_{-i})$. And since $\{\hat{C}_i\}_{i\in L}$ satisfy irrelevance of rejected contracts, $\hat{C}_i(Y_i|Z^*_{-i}) = \hat{C}_i(Z_i|Z^*_{-i})$ for each $i \in L$. Then we have

$$Y^* = \bigcup_{i\in L} \hat{C}_i(Y_i|Y^*_{-i}) = \bigcup_{i\in L} \hat{C}_i(Y_i|Z^*_{-i}) = \bigcup_{i\in L} \hat{C}_i(Z_i|Z^*_{-i}) = Z^*.$$

Specification of $\{C_i, \mu_i\}_{i\in L}$. For each $i \in L$ and $Y \subseteq X$, let

$$C_i(Y_i|Y_{-i}) = Y^*_i, \qquad \mu_i(Y) = Y^*_{-i}.$$

Claim 5. For each $Y \subseteq X$ and $i \in L$, $Y^*_i = \hat{C}_i(Y_i|Y^*_{-i})$. Since the market is two-sided, X_i and X_j are disjoint for each $i, j \in L$ with $i \neq j$. Since Y^* is a fixed point of G_Y, $Y^* = \bigcup_{i\in L} \hat{C}_i(Y_i|Y^*_{-i})$. Then $\hat{C}_i(Y_i|Y^*_{-i}) = X_i \cap \left(\bigcup_{i\in L} \hat{C}_i(Y_i|Y^*_{-i})\right) = X_i \cap Y^* = Y^*_i$.

Claim 6. $\{\mu_i\}_{i\in L}$ are correct given $\{C_i\}_{i\in L}$. By definition, for each $i \in L$,

$$C_{L-i}(Y) = \left(\bigcup_{j\in L\setminus\{i\}} C_j(Y_j|Y_{-j})\right) = Y^*_{-i} = \mu_i(Y).$$

The claim follows.

Claim 7. **$\{C_i\}_{i\in L}$ are optimal given $\{\mu_i\}_{i\in L}$.** By Claim 5, for each $Y \subseteq X$ and $i \in L$,

$$C_i(Y_i|Y_{-i}) = Y_i^* = \hat{C}_i(Y_i|Y_{-i}^*) = \hat{C}_i(Y_i|\mu_i(Y)) = \arg\max_{S\subseteq Y_i} u_i(S \cup \mu_i(Y)).$$

The claim follows.

Claim 8. **$\{\mu_i\}_{i\in L}$ are cross-set consistent given $\{C_i\}_{i\in L}$.** Observe that since, by definition, $\bigcup_{i\in L} X_i = X$, $C_L(Y) = \bigcup_{i\in L} Y_i^* = Y^*$ for each $Y \subseteq X$. Then by Claim 4, $C_L(Z) \subseteq Y \subseteq Z$ implies $C_L(Y) = C_L(Z)$. Since $\{\mu_i\}_{i\in L}$ are correct given $\{C_i\}_{i\in L}$ (Claim 6), the claim follows from Lemma 2.

Claim 9. **$\{C_i\}_{i\in L}$ are substitutable.** Suppose $Y \subseteq Z$. From Claim 2, $Y^* \preceq_L Z^*$. Then since $\{\hat{C}_i\}_{i\in L}$ satisfy standard substitutes and monotone externalities,
$\hat{R}_i(Y_i|Y_{-i}^*) \subseteq \hat{R}_i(Z_i|Z_{-i}^*)$ for each $i \in L$. Then for each $i \in L$,

$$\hat{C}_i(Y_i|Y_{-i}^*) = Y_i \setminus \hat{R}_i(Y_i|Y_{-i}^*) \supseteq Y_i \setminus \hat{R}_i(Z_i|Z_{-i}^*) = \hat{C}_i(Z_i|Z_{-i}^*) \cap Y_i.$$

Then by Claim 5,

$$C_i(Y_i|Y_{-i}) = Y_i^* = \hat{C}_i(Y_i|Y_{-i}^*) \supseteq \hat{C}_i(Z_i|Z_{-i}^*) \cap Y_i = Z_i^* \cap Y_i = C_i(Z_i|Z_{-i}) \cap Y_i.$$

Hence,

$$R_i(Y_i|Y_{-i}) = Y_i \setminus C_i(Y_i|Y_{-i}) \subseteq Y_i \setminus C_i(Z_i|Z_{-i}) \subseteq Z_i \setminus C_i(Z_i|Z_{-i}) = R_i(Z_i|Z_{-i}),$$

as desired. □